Solutions with Vortices of a Semi-Stiff Boundary Value Problem for the Ginzburg-Landau Equation

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Abstract

We study solutions of the 2D Ginzburg-Landau equation

$$-\Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0$$

subject to "semi-stiff" boundary conditions: the Dirichlet condition for the modulus, $|u| = 1$, and the homogeneous Neumann condition for the phase. The principal result of this work shows there are stable solutions of this problem with zeros (vortices), which are located near the boundary and have bounded energy in the limit of small ε. For the Dirichlet bondary condition ("stiff" problem), the existence of stable solutions with vortices, whose energy blows up as $\varepsilon \to 0$, is well known. By contrast, stable solutions with vortices are not established in the case of the homogeneous Neumann ("soft") boundary condition.

In this work, we develop a variational method which allows one to construct local minimizers of the corresponding Ginzburg-Landau energy functional. We introduce an approximate bulk degree as the key ingredient of this method, and, unlike the standard degree over the curve, it is preserved in the weak H^1-limit.

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1 Introduction and Main Results

In this work, we study solutions of the Ginzburg-Landau (GL) equation

$$-\Delta u + \frac{1}{\varepsilon^2} u(|u|^2 - 1) = 0 \quad \text{in} \quad A, \tag{1.1}$$

where ε is a positive parameter (the inverse of the GL parameter $\kappa = 1/\varepsilon$), u is a complex-valued ($\mathbb{R}^2$-valued) map, and A is a smooth, bounded, multiply connected domain in $\mathbb{R}^2$. For simplicity, hereafter we assume A is an annular type (doubly connected) domain of the form $A = \Omega \setminus \overline{\omega}$, where Ω and ω are simply connected smooth domains and $\overline{\omega} \subset \Omega \subset \mathbb{R}^2$.

Equation (1.1) is the Euler-Lagrange PDE corresponding to the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_A |\nabla u|^2 \mathrm{d}x + \frac{1}{4\varepsilon^2}\int_A (|u|^2 - 1)^2 \mathrm{d}x. \tag{1.2}$$

Equations of this type arise, e.g., in models of superconductivity and superfluidity. Additionally, (1.1) is viewed as a complex-valued version of the Allen-Cahn model for phase transitions [32].

Solutions of (1.1) subject to the Dirichlet boundary condition, $u = g$ on ∂A with fixed S^1-valued boundary data g, have been extensively studied in the past decade. Special attention has been paid to solutions with isolated zeros (vortices). In contrast with the Dirichlet problem, in the case of the homogeneous Neumann boundary condition, solutions are typically vortexless; in particular, stable solutions with vortices have not been established.

This work is devoted to solutions of (1.1) subject to the "semi-stiff" boundary conditions

$$|u| = 1 \text{ and } u \times \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial A. \tag{1.3}$$

These boundary conditions are intermediate between Dirichlet and Neumann in the following sense: any solution $u \in H^1(A;\mathbb{R}^2)$ of (1.1, 1.3) is sufficiently regular [11], so it can be written as $u = |u|\mathrm{e}^{i\psi}$ (locally) near the boundary. Then (1.3) means the Dirichlet boundary condition is prescribed for the modulus, $|u| = 1$ on ∂A, and the Neumann condition is prescribed for the phase, $\frac{\partial \phi}{\partial \nu} = 0$ on ∂A.

Problem (1.1, 1.3) is equivalent to finding critical points of the energy functional (1.2) in the space

$$\mathcal{J} = \{u \in H^1(A;\mathbb{R}^2);\ |u| = 1 \text{ a.e. on } \partial A\}. \tag{1.4}$$

Our main objective is to study the existence of *stable* solutions of (1.1, 1.3) with vortices. Since the problem is time independent, stable solutions are defined as (local) minimizers of (1.2) in $\mathcal{J}$. In other words, we are interested in whether the model (1.1, 1.3) stabilizes vortices similarly to Dirichlet problem or does not stabilize vortices analogously to Neumann problem. The boundary conditions (1.3) are not well studied, and this work, along with studies [7, 11–13, 21] reveals their distinct features, described later in the introduction.

Let us briefly review the existing results for the Dirichlet and Neumann boundary value problems for equation (1.1). The first results on the existence of stable solutions with vortices for Dirichlet problem were obtained in [19, 20]. Stable solutions of (1.1) with vortices were obtained and studied in [14] for star-shaped domains and prescribed S^1-valued boundary data with nonzero topological degree. In [14], the limiting locations (as $\varepsilon \to 0$) of vortices of globally minimizing and other solutions (if they exist) are described by means of a *renormalized energy*. Subsequently, these results were generalized for multiply connected domains in [34]. The existence of locally minimizing and minmax solutions was established first in [28] and [29], then in more detail and generality in [27] (see also [4, 17]). We refer the reader to [9], and references therein, for the Dirichlet problem's various results. As previously mentioned, only vortexless stable solutions of (1.1) with the homogeneous Neumann boundary condition in 2D are known. Moreover, all locally minimizing solutions are constant maps if A is convex [22], or simply connected and ε is small [33]. The existence of nonconstant (but vortexless) locally minimizing solutions is established in [23] and [3]. In the recent work [17], a general result for the existence of (nonminimizing) solutions with vortices was obtained. Similarly to the Dirichlet problem, these solutions with vortices have energy that blows up as $\varepsilon \to 0$.

Equation (1.1) (functional (1.2)) is usually referred to as a simplified GL model (without magnetic field). There is a large body of mathematical literature on the general GL model with a magnetic field (e.g., [2, 5, 24, 30, 31]). Since (1.1) is obtained from the general GL energy by setting the magnetic field to zero, it describes persistent currents in a 2D cross-section of a cylindrical superconductor (or in a 2D film). It was observed in [14], the degree of the boundary data on connected components of ∂A creates the same type of "quantized vortices" as a magnetic field in type II superconductors or as angular rotation in superfluids. Despite a relatively

simple form of equation (1.1), it leads to a deep analysis of properties of its solutions similar to other fundamental PDE's in mathematical physics.

The boundary conditions (1.3) model, e.g., the surface of a superconductor coated with a high temperature superconducting thin film [6]. Generating a mathematical model of persistent currents in such a superconductor, then amounts to finding critical points of functional (1.2) in the space $\mathcal{J}$, when $u = |u|e^{i\psi}$ on the boundary and $|u| = 1$, while the phase ψ is "free".

Boundary conditions (1.3) appeared in recent studies [7, 12, 21] of the minimization problem for the GL functional (1.2) among maps from $\mathcal{J}$ with prescribed degrees on the connected components of the boundary. The minimization of the energy (1.2) in $\mathcal{J}$ produces only constant solutions of (1.1, 1.3), similar to the case of the Neumann problem, which corresponds to finding critical points of (1.2) in the entire space $H^1(A; \mathbb{R}^2)$. An obvious way of producing critical points with vortices is to impose two different degrees $q \neq p$ on $\partial\Omega$ and $\partial\omega$. That is, to consider the minimization of $E_\varepsilon(u)$ in the set $\mathcal{J}_{pq} \subset \mathcal{J}$, where

$$\mathcal{J}_{pq} := \{u \in \mathcal{J};\ \deg(u, \partial\omega) = p, \deg(u, \partial\Omega) = q.\} \tag{1.5}$$

Recall that the degree (winding number) of a map $u \in H^{1/2}(\gamma, S^1)$ on γ (where γ is either $\partial\omega$ or $\partial\Omega$) is an integer given by the classical formula (cf., e.g., [8])

$$\deg(u, \gamma) = \frac{1}{2\pi} \int_\gamma u \times \frac{\partial u}{\partial \tau} \mathrm{d}\tau, \tag{1.6}$$

where the integral is understood via $H^{1/2} - H^{-1/2}$ duality, and $\frac{\partial}{\partial \tau}$ is the tangential derivative with respect to the counterclockwise orientation of γ. (Throughout the paper we assume the same orientation of $\partial\omega$ and $\partial\Omega$.) Note that $\mathcal{J}_{pq}$ are connected components of $\mathcal{J}$ (see [8]).

Simple topological considerations imply that critical points from $\mathcal{J}_{pq}$ must have at least $|p - q|$ (with multiplicity) vortices. We emphasize that the existence of such critical points is far from obvious. For example (see Section 2), there are no global minimizers of $E_\varepsilon(u)$ in $\mathcal{J}_{01}$ and the weak limits of minimizing sequences do not belong to $\mathcal{J}_{01}$. This simple example illustrates an important property of the sets $\mathcal{J}_{pq}$, which is crucial for our consideration: *these sets are not weakly H^1-closed*, since the degree at the boundary may change in the limit. On the other hand, the results

of this work show that when $p = q$ and there is no topological reason for vortices to appear, local minimizers typically do have vortices.

As mentioned above, the vortex structure of solutions of (1.1) with Dirichlet and Neumann boundary conditions is well studied. In contrast, only vortexless solutions of the semi-stiff problem (1.1), (1.3) were found [21], [12]. In [12], it was shown that minimizing sequences for the corresponding minimization problem develop a novel type of so-called "near-boundary" vortices, which approach the boundary and have finite GL energy in the limit of small ε (due to the ghost vortices, see Appendix A). However, such minimizing sequences do not converge to actual minimizers [13]. These studies lead to the natural question of whether there exist true solutions of (1.1, 1.3) with near-boundary vortices. Unlike the minimizing sequences such solutions may model observable states of a physical system (e.g., persistent currents with vortices and superfluids between rotating cylinders [18]). The following theorem, which is the main result of this work, provides the answer to this question.

Theorem 1 (Existence of solutions with vortices of problem (1.1, 1.3))**.** *For any integer $M > 0$, there exist at least M distinct stable solutions of (1.1, 1.3) with (nearboundary) vortices when $\varepsilon < \varepsilon_1$ ($\varepsilon_1 = \varepsilon_1(M) > 0$). The vortices of these solutions are at distance $o(\varepsilon)$ from the boundary and have bounded GL energy in the limit $\varepsilon \to 0$. The solutions are stable in the sense that they are (local) minimizers of (1.2) in $\mathcal{J}$.*

To construct local minimizers of (1.2) in $\mathcal{J}$, we represent $\mathcal{J}$ as the union of subsets $\mathcal{J}_{pq}^{(d)}$ (defined in (1.8) below), $\mathcal{J} = \cup_{p,q,d\in\mathbb{Z}}\mathcal{J}_{pq}^{(d)}$, and study the existence of global minimizers in $\mathcal{J}_{pq}^{(d)}$. Furthermore, we show that each minimizer lies in $\mathcal{J}_{pq}^{(d)}$ with its open neighborhood. Therefore, the minimizers in $\mathcal{J}_{pq}^{(d)}$ are distinct local minimizers in $\mathcal{J}$.

Thus, the construction of solutions of (1.1, 1.3) is based on the study of the following constrained minimization problem:

$$m_\varepsilon(p, q, d) := \inf\{E_\varepsilon(u);\, u \in \mathcal{J}_{pq}^{(d)}\}, \tag{1.7}$$

where

$$\mathcal{J}_{pq}^{(d)} = \{u \in \mathcal{J}_{pq}; \operatorname{abdeg}(u) \in [d - 1/2, d + 1/2]\}, \tag{1.8}$$

p,q and d are given integers, and $\operatorname{abdeg}(u)$ is the *approximate bulk degree*, introduced as follows. Consider the boundary value problem

$$\begin{cases} \Delta V = 0 \quad \text{in } A \\ V = 1 \quad \text{on } \partial\Omega \\ V = 0 \quad \text{on } \partial\omega. \end{cases} \tag{1.9}$$

Introduce $\operatorname{abdeg}(\,\cdot\,):\ H^1(A;\mathbb{R}^2) \to \mathbb{R}$ by the formula

$$\operatorname{abdeg}(u) = \frac{1}{2\pi}\int_A u \times (\partial_{x_1}V\,\partial_{x_2}u - \partial_{x_2}V\,\partial_{x_1}u)\,\mathrm{d}x, \tag{1.10}$$

where V solves (1.9). In the particular case where A is a circular annulus, $A_{R_1R_2} = \{x; R_1 < |x| < R_2\}$, $\operatorname{abdeg}(u)$ is expressed by

$$\operatorname{abdeg}(u) = \frac{1}{\log(R_2/R_1)}\int_{R_1}^{R_2}\Big(\frac{1}{2\pi}\int_{|x|=\xi} u \times \frac{\partial u}{\partial \tau}\mathrm{d}s\Big)\frac{\mathrm{d}\xi}{\xi}. \tag{1.11}$$

For S^1-valued maps, $\operatorname{abdeg}(u)$ becomes integer valued and representation (1.11) clarifies its interpretation as an average value of the standard degree. The definition (1.10) is motivated by the following intuitive consideration: represent the standard degree over the boundary $\partial\Omega$ via a "bulk" integral over the area of A for S^1-valued maps and notice that if $E_\varepsilon(u) \le \Lambda$ for some finite Λ and sufficiently small ε, then u is "almost" S^1-valued.

It was observed in [3] that for S^1-valued maps in an annulus A, one can define the topological degree $\deg(u, A)$ that classifies maps $u \in H^1(A; S^1)$ according to their 1-homotopy type [35] (1-homotopy type is completely determined by the degree of the restriction to a nontrivial contour). This definition was relaxed in [3] for maps that are not necessarily S^1-valued by considering $u/|u|$ in a subdomain $A_u \subset A$, which is obtained by removing neighborhoods of the boundary ∂A and zeros (vortices) of u. Definition (1.11) does not require the removal of vortices from A, and $\operatorname{abdeg}(u)$ is obtained by a simple formula (unlike $\deg(u, A)$ in [3], where the domain of integration depends on u). Note that in general $\operatorname{abdeg}(u)$ is not an integer. The most important fact for our consideration is that $\operatorname{abdeg}(u)$ is continuous with respect to weak H^1-convergence, unlike the standard degree in (1.6) (this issue for $\deg(u, A)$ was not addressed in [3]).

The minimization in problem (1.7) is taken over $\mathcal{J}^{(d)}_{pq}$, which is not an open set, and therefore minimizers of (1.7) (if they exist) are not necessarily local minimizers

of (1.2) in $\mathcal{J}$. Indeed, while $\mathcal{J}_{pq}$ is an open subset of $\mathcal{J}$ (hereafter we assume the topology and convergence in $\mathcal{J}$ to be the strong H^1 unless otherwise is specified), the constraint $\mathrm{abdeg}(u) \in [d-1/2, d+1/2]$ defines a closed set with respect to both strong and weak H^1-convergences. However, if we further consider a subset of maps with bounded energy and choose ε small enough, then the constraint $\mathrm{abdeg}(u) \in [d-1/2, d+1/2]$ becomes open thanks to the following proposition:

Proposition 2. *Fix $\Lambda > 0$. There exists $\varepsilon_0 = \varepsilon_0(\Lambda) > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then for any integer d and any $u \in H^1(A;\mathbb{R}^2)$ satisfying $E_\varepsilon(u) \le \Lambda$ the closed constraint* $\mathrm{abdeg}(u) \in [d-1/2, d+1/2]$ *is equivalent to an open one that is,*

$$d - 1/2 \le \mathrm{abdeg}(u) \le d + 1/2 \iff d - 1/2 < \mathrm{abdeg}(u) < d + 1/2. \qquad (1.12)$$

The following theorem is the main tool in proving existence of local minimizers.

Theorem 3 (Existence of minimizers of the constrained problem)**.** *For any integers p, q and $d > 0$ ($d < 0$) with $d \ge \max\{p,q\}$ ($d \le \min\{p,q\}$) there exists $\varepsilon_1 = \varepsilon_1(p,q,d) > 0$ such that the infimum in (1.7) is always attained, when $\varepsilon < \varepsilon_1$. Moreover*

$$m_\varepsilon(p,q,d) \le I_0(d,A) + \pi(|d-p| + |d-q|), \qquad (1.13)$$

where

$$I_0(d,A) = \min\left\{ \frac{1}{2}\int_A |\nabla u|^2 \mathrm{d}x,\ u \in H^1(A,S^1) \cap \mathcal{J}_{dd} \right\}. \qquad (1.14)$$

The value $I_0(d,A)$ is expressed by $I_0(d,A) = 2(\pi d)^2/\mathrm{cap}(A)$ via the H^1-capacity $\mathrm{cap}(A)$ *of the domain A.*

The key difficulty is to establish the attainability of the infimum in (1.7), which is highly nontrivial since the degree on $\partial\Omega$ and $\partial\omega$ is not preserved in the weak H^1-limit [12, 13]. We show solutions of (1.1, 1.3), which are minimizers of (1.7) (local minimizers of (1.2) in $\mathcal{J}$) with $p \ne d$ and any q (or $q \ne d$ and any p) must have vortices. For fixed ε, these vortices are located at a positive distance from ∂A and approach ∂A as $\varepsilon \to 0$.

Without loss of generality, throughout this work we always assume that $d > 0$ (otherwise one can reverse the orientation of $\mathbb{R}^2$).

Theorem 1 follows from Theorem 3 and Proposition 2. The asymptotic behavior of the local minimizers is established in

Theorem 4 (Asymptotic behavior of minimizers and their energies)**.** *Assume that the integers p, q and d satisfy the assumptions of Theorem 3. Then as $\varepsilon \to 0$ minimizers of (1.7) converge weakly in $H^1(A)$, up to a subsequence, to a harmonic map u which minimizes (1.14). Additionally,*

$$E_\varepsilon(u_\varepsilon) = I_0(d, A) + \pi(|d-p| + |d-q|) + o(1), \quad \textit{as } \varepsilon \to 0, \textit{ and} \tag{1.15}$$

$$E_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_A |\nabla u_\varepsilon|^2 \mathrm{d}x + o(1), \quad \textit{as } \varepsilon \to 0. \tag{1.16}$$

In particular, it follows from (1.15, 1.16) that there is no strong convergence of minimizers of (1.7) in $H^1(A)$ as $\varepsilon \to 0$ unless $p = q = d$.

Next, we summarize the distinct features of the GL boundary value problem with semi-stiff boundary conditions. The first interesting feature is the existence of solutions with a new type of vortices called *near-boundary vortices*. Unlike the inner vortices, whose energy blows up at the rate of $|\log \varepsilon|$, the energy of near-boundary vortices is bounded as $\varepsilon \to 0$ and they are located at a distance $o(\varepsilon)$ from the boundary.

Secondly, the semi-stiff boundary conditions result in a lack of compactness. Namely, as of now, the only way to find nonconstant minimizers is by searching for minimizers in subsets $\mathcal{J}_{pq}^{(d)} \subset \mathcal{J}$. These subsets, however, are not weakly H^1-closed and therefore a weak H^1-limit of a minimizing sequence $(u^{(k)}) \in \mathcal{J}_{pq}^{(d)}$ may not lie in $\mathcal{J}_{pq}^{(d)}$, but rather in $\mathcal{J}_{p'q'}^{(d)}$ with $p' \neq p$ or (and) $q' \neq p$. Theorem 3 shows that if $d > 0$ and $d \geq \max\{p, q\}$, then (for small ε) any weak H^1-limit of a *minimizing sequence* $(u^{(k)}) \subset \mathcal{J}_{pq}^{(d)}$ always belongs to $\mathcal{J}_{pq}^{(d)}$, despite the lack of weak H^1-closeness of $\mathcal{J}_{pq}^{(d)}$. In contrast, if $d \geq 0$ and $d < \max\{p, q\}$ we have

Conjecture 5. Let $d \geq 0$, $d < \max\{p, q\}$ (or $d \leq 0$, $d > \min\{p, q\}$) and let u be a weak limit of a minimizing sequence for problem (1.7) (such a minimizing sequence exists and bound (1.13) holds for any integer p, q, d, see Appendix B). Then $u \notin \mathcal{J}_{pq}^{(d)}$ when ε is sufficiently small.

In the simplest case, when $d = 0$ and either $p = 1$ and $q = 0$ or $p = 0$ and $q = 1$, this conjecture is demonstrated by an argument quite similar to the nonexistence proof in [11] for simply connected domains (see Sec.2 below). A more interesting example, which supports the above conjecture, follows from the previously studied (global) minimization problem $\tilde{m}_\varepsilon = \inf\{E_\varepsilon(u),\, u \in \mathcal{J}_{11}\}$. It was shown in [11, 13]

that if $\mathrm{cap}(A) \geq \pi$ (subcritical/critical cases), then $\tilde{m}_\varepsilon$ is always attained, whereas if $\mathrm{cap}(A) < \pi$ (supercritical case), then $\tilde{m}_\varepsilon$ is never attained for small ε. One can see elements of minimizing sequences lie in $\mathcal{J}_{11}^{(1)}$ in subcritical/critical cases and in $\mathcal{J}_{11}^{(0)}$ in the supercritical case. Moreover, the nonexistence of minimizers in problem (1.7) for $d = 0$, $p = q = 1$ and small ε holds for any doubly connected domain (with any capacity). (For $\mathrm{cap}(A) < \pi$ the proof is presented in [13], this proof can be easily generalized for $\mathrm{cap}(A) \geq \pi$.)

We conclude the introduction by outlining the scheme of the proof of Theorem 3, which employs a comparison argument. Fix an integer $d > 0$. First, we establish the existence of minimizers in problem (1.7) for $p = q = d$ by using the so-called Price Lemma [12] (see Lemma 9 below), the uniform lower energy bound from Lemma 16 and the upper bound from Lemma 14, which is obtained by considering S^1-valued testing maps. We show these minimizers (which belong to $\mathcal{J}_{dd}^{(d)}$) are vortexless. Next, we argue by induction on the parameter $æ(p, q) = |d - p| + |d - q|$. This parameter is naturally associated with the number of vortices– for example, for the above minimizers in $\mathcal{J}_{dd}^{(d)}$, we have $æ(d, d) = 0$. Given an integer $K \geq 0$, we assume the existence of minimizers in problem (1.7) for p, q such that, $æ(p, q) \leq K$ and $p \leq d$, $q \leq d$ (the induction hypothesis) and prove the existence of minimizers for p, q such that $æ(p, q) = K + 1$ and $p \leq d$, $q \leq d$. The first step in the induction procedure (when $K = 0$) is shown in Section 5. The key technical point there is to construct a testing map $v \in \mathcal{J}_{d(d-1)}^{(d)}$ such that

$$E_\varepsilon(v) < E_\varepsilon(u_0) + \pi, \tag{1.17}$$

where u_0 is a minimizer of (1.2) in $\mathcal{J}_{dd}^{(d)}$. This map v is constructed by using the minimizer u_0 and Möbius conformal maps (Blashke factor [15]) on the unit disk with a prescribed single zero near the boundary. Then, given a minimizing sequence $(u^{(k)}) \subset \mathcal{J}_{d(d-1)}^{(d)}$ of problem (1.7) for $p = d$, $q = d-1$, we have, by (2.7) from Lemma [9] and (1.17),

$$E_\varepsilon(u)+\pi(|d-\deg(u, \partial\omega)|+|d-1-\deg(u, \partial\Omega)|) \leq \lim_{k\to\infty} E_\varepsilon(u^{(k)}) < E_\varepsilon(u_0)+\pi, \tag{1.18}$$

where u is a weak H^1-limit of $(u^{(k)})$ (possibly a subsequence). Then we estimate the left hand side of (1.18) by the lower energy bound from Lemma 16 and the right

hand side of (1.18) by the upper bound from Lemma 14, this yields

$$I_0(d,A)+\pi(2|d-\deg(u,\partial\omega)|+|d-1-\deg(u,\partial\Omega)|+|d-\deg(u,\partial\Omega)|) < I_0(d,A)+\frac{3}{2}\pi. \tag{1.19}$$

This implies that $\deg(u,\partial\omega)=d$, and either $\deg(u,\partial\Omega)=d-1$ or $\deg(u,\partial\Omega)=d$. In view of (1.18), the only possible case is actually $\deg(u,\partial\Omega)=d-1$ since, otherwise, $u\in\mathcal{J}^{(d)}_{dd}$ and therefore $E_\varepsilon(u)\geq E_\varepsilon(u_0)$ which contradicts (1.18). Thus $u\in\mathcal{J}^{(d)}_{d(d-1)}$ and is a minimizer in $\mathcal{J}^{(d)}_{d(d-1)}$. The proof of existence of minimizers for $p=d-1$, $q=d$ is quite similar. So we have shown that the existence of minimizers for $æ(p,q)=0$ implies the existence of minimizers for $æ(p,q)=1$. In the general case, when passing from $æ(p,q)\leq K$ to $æ(p,q)\leq K+1$ in problem (1.7), we use the same idea but it is technically much more involved. It requires the asymptotic analysis as $\varepsilon\to 0$ of minimizers u_{pq} of (1.7) with $æ(p,q)=K$, which is carried out in Section 6. Based on the result of this asymptotic analysis, we construct testing maps $v\in\mathcal{J}^{(d)}_{p'q'}$ ($p'=p$, $q'=q-1$ or $p'=p$, $q'=q-1$), such that $E_\varepsilon(v)<E_\varepsilon(u_{pq})+\pi$.

2 Preliminaries

Throughout the paper we use the following notations.

- The vectors $a=(a_1,a_2)$ are identified with complex numbers $a=a_1+ia_2$.
- $a\cdot b$ stands for the scalar product $a\cdot b=a_1b_1+a_2b_2=\frac{1}{2}(a\bar{b}+\bar{a}b)$.
- $a\times b$ stands for the vector product $a\times b=a_1b_2-a_2b_1=\frac{i}{2}(a\bar{b}-\bar{a}b)$.
- The orientation of simple (without self intersecting) curves in $\mathbb{R}^2$ (in particular $\partial\omega$ and $\partial\Omega$) is assumed counterclockwise. If $\mathcal{L}$ is such a curve, τ stands for the unit tangent vector pointing in the sense of the above mentioned orientation on $\mathcal{L}$, ν is the unit normal vector such that (ν,τ) is direct.
- If h is a scalar function, then $\nabla^\perp h=(-\partial_{x_2}h,\partial_{x_1}h)$.

2.1 Properties of solutions from problem (1.1, 1.3)

As shown in [11] by a bootstrap argument, any solution $u\in H^1(A;\mathbb{R}^2)$ of problem (1.1, 1.3) is sufficiently regular (e.g., $u\in C^2(\overline{A})$ if A has a C^2 boundary). By the

maximum principle we also have

Lemma 6. *The function $\rho(x) = |u(x)|$ satisfies $\rho \leq 1$ in A.*

Locally, away from its zeros, u can be written as $u = \rho e^{i\phi}$ with real-valued phase ϕ. We also will frequently make use of the current potential h related to the solution u of (1.1, 1.3) by

$$\begin{cases} \nabla^{\perp} h = (u \times \partial_{x_1} u, u \times \partial_{x_2} u) & \text{in } A \\ h = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Unlike the phase ϕ, the function h is defined globally on A, and

$$\nabla h = -\rho^2 \nabla^{\perp} \phi \quad \text{when } \rho > 0. \tag{2.2}$$

The existence of the unique solution of system (2.1) and its elementary properties are established in the following

Lemma 7. *There exists the unique solution h of the system (2.1) and $h = \mathrm{Const}$ on $\partial\omega$, moreover*

$$\Delta h = 2\partial_{x_1} u \times \partial_{x_2} u \quad \textit{in } A, \tag{2.3}$$

$$\mathrm{div}(\frac{1}{\rho^2}\nabla h) = 0 \quad \textit{when } \rho > 0. \tag{2.4}$$

Proof. The vector field $F = (u \times \partial_{x_1} u, u \times \partial_{x_2} u)$ is divergence free. Indeed, since u is a smooth solution of (1.1), we have $\mathrm{div} F = u \times \Delta u = 0$ in A. It follows that for any simply connected domain $W \subset A$ there is a unique (up to an additive constant) function Φ solving $\nabla^{\perp}\Phi = F$ in W (this is well known Poincaré's lemma) . Such a local solution Φ can be extended to a (possibly multi-valued) solution on A. Thanks to the fact that u satisfies (1.3) we have $\frac{\partial \Phi}{\partial \tau} = -F \cdot \nu = 0$ on ∂A, i.e. Φ takes constant values on every connected component of the boundary. This means Φ is actually a single valued function. Then, $h(x) = \Phi(x) - \Phi(\partial\Omega) + 1$ is the unique solution of (2.1).

The verification of (2.3) is straightforward; (2.4) follows directly from (2.2). $\square$

In what follows, we also use the following result, which is valid for any solution of the GL equation (1.1) (not necessarily satisfying (1.3)).

Lemma 8 ([26]). *Let u be a solutions of the GL equation (1.1) such that $|u| \le 1$, $E_\varepsilon(u) \le \Lambda$, where Λ is independent of ε. Then*

$$1 - |u(x)|^2 \le \frac{\varepsilon^2 C}{\operatorname{dist}^2(x, \partial A)} \tag{2.5}$$

and

$$|D^k u(x)| \le \frac{C_k}{\operatorname{dist}^k(x, \partial A)}. \tag{2.6}$$

where C, C_k are independent of ε.

2.2 Minimization among maps of $\mathcal{J}$ with prescribed degrees

Any minimizer of (1.2) over the set $\mathcal{J}_{pq}$ with prescribed integer degrees p and q is clearly a solution of (1.1), (1.3). However, the existence of minimizers is a nontrivial problem. In [7, 11–13, 21] the minimization problem for the Ginzburg-Landau functional (1.2) in $\mathcal{J}_{11}$ was considered. In the case when A is a circular annulus, it was observed in [7] that minimizers, if they exist, brake the symmetry when the ratio of the outer and inner radii of the annulus exceeds certain threshold. By contrast, in the case when this ratio is sufficiently close to 1, the existence of a unique minimizer and its symmetry is shown in [21]. The techniques in both [7] and [21] relied on the circular symmetry of the domain. A more general approach based on the Price Lemma was proposed in [11],[12].

Lemma 9. *[12] Let $(u^{(k)} \in \mathcal{J}_{pq})$ be a sequence that converges to u weakly in $H^1(A, \mathbb{R}^2)$. Then*

$$\liminf_k \frac{1}{2}\int_A |\nabla u^{(k)}|^2 dx \ge \frac{1}{2}\int_A |\nabla u|^2 dx + \pi(|p - \deg(u, \partial\omega)| + |q - \deg(u, \partial\Omega)|),$$

or, equivalently (by Sobolev embeddings),

$$\liminf_k E_\varepsilon(u^{(k)}) \ge E_\varepsilon(u) + \pi(|p - \deg(u, \partial\omega)| + |q - \deg(u, \partial\Omega)|). \tag{2.7}$$

This result is also of prime importance in this work. With the help of Lemma 9, it was shown in [12] that the infimum of (1.2) in $\mathcal{J}_{11}$ is always attained when $\operatorname{cap}(A) \ge \pi$. It was also conjectured in [12] that when $\operatorname{cap}(A) < \pi$ and ε is sufficiently small the weak limit of any minimizing sequence is not in the class of admissible

maps, i.e. the global minimizer does not exist. In [13] this nonexistence conjecture was proved by a contradiction argument based on explicit energy bounds.

While the existence/nonexistence of minimizers in $\mathcal{J}_{pq}$ for $p = q = 1$ is nontrivial and the answer depends on $\mathrm{cap}(A)$ and also on ε, the case $p = 0$, $q = 1$ ($p = 1$, $q = 0$) is simple. Arguing as in [11] we can show that $\inf\{E_\varepsilon(u); u \in \mathcal{J}_{01}\}$ is never attained. Really, we have

$$\frac{1}{2}\int_A |\nabla u|^2 \mathrm{d}x \geq \left|\int_A \partial_{x_1} u \times \partial_{x_2} u \mathrm{d}x\right| = \pi|\deg(u, \partial\Omega) - \deg(u, \partial\omega)| = \pi$$

whenever $u \in \mathcal{J}_{01}$. On the other hand, by constructing explicit sequence in the spirit of [7] (see also [11]), we have $\inf\{E_\varepsilon(u); u \in \mathcal{J}_{01}\} = \pi$. Thus, if there exists a minimizer $u \in \mathcal{J}_{01}$ then $u \in H^1(A; S^1)$ and u solves the GL equation (1.1). This is impossible unless u is a constant map, and then $u \notin \mathcal{J}_{01}$.

3 Properties of the approximate bulk degree

The degree of restriction of maps from $H^1(A, S^1)$ to any smooth closed curve, in particular $\partial\Omega$, is preserved by the weak H^1-convergence. This result follows from [35], or can be shown directly by using integration by parts as below in (3.2) (note that, for any S^1-valued map u, $\deg(u, \partial\Omega) = \deg(u, \partial\omega) = \deg(u, \mathcal{L})$, where $\mathcal{L}$ is an arbitrary smooth simple curve in A enclosing ω). Thus we have the decomposition

$$H^1(A, S^1) = \bigcup_{d\in\mathbb{Z}} \{u \in H^1(A, S^1),\ \deg(u, \partial\Omega) = d\} \tag{3.1}$$

by disjoint sets, each of them being closed in weak H^1-topology of $H^1(A, S^1)$.

Fix $\Lambda > 0$. In this section we consider maps $u \in H^1(A, \mathbb{R}^2)$ in the level set $E_\varepsilon^\Lambda = \{u; E_\varepsilon(u) \leq \Lambda\}$. We show that the approximate bulk degree $\mathrm{abdeg}(u)$ classifies maps $u \in E_\varepsilon^\Lambda$ similarly to the above classification (3.1) of S^1-valued maps. The basic properties of $\mathrm{abdeg}(u)$ we demonstrate are

a) $\mathrm{abdeg}(u, A) = \deg(u, \partial\Omega)$ if $u \in H^1(A, S^1)$,

b) $|\mathrm{abdeg}(u) - \mathrm{abdeg}(v)| \leq \frac{2}{\pi}\|V\|_{C^1(A)}\Lambda^{1/2}\|u - v\|_{L^2(A)}$ if $u, v \in E_\varepsilon^\Lambda$.

The first property follows directly from the definition (1.10) of $\mathrm{abdeg}(u)$. Really, integrating by parts in (1.10), we get

$$\mathrm{abdeg}(u) = \frac{1}{2\pi}\int_{\partial\Omega} u \times \frac{\partial u}{\partial\tau}\mathrm{d}s - \frac{1}{\pi}\int_A \partial_{x_1} u \times \partial_{x_2} u\, V \mathrm{d}x = \deg(u, \partial\Omega), \tag{3.2}$$

for any $u \in H^1(A, S^1)$ ($\partial_{x_1} u \times \partial_{x_2} u = 0$ a.e. in A since $|u| = 1$ a.e.). The property b) of $\mathrm{abdeg}(u)$ is proved in the following

Lemma 10. *For any $u, v \in H^1(A; \mathbb{R}^2)$ we have*

$$|\mathrm{abdeg}(u) - \mathrm{abdeg}(v)| \leq \frac{1}{\pi} \|V\|_{C^1(A)} ((E_\varepsilon(u))^{1/2} + (E_\varepsilon(v))^{1/2}) \|u - v\|_{L^2(A)}.$$

Proof. We have, integrating by parts,

$$\begin{aligned} 2\pi(\mathrm{abdeg}(u) - \mathrm{abdeg}(v)) &= \int_A (u - v) \times (\partial_{x_2} u \, \partial_{x_1} V - \partial_{x_1} u \, \partial_{x_2} V) \mathrm{d}x \\ &\quad + \int_A v \times (\partial_{x_2} (u - v) \, \partial_{x_1} V - \partial_{x_1} (u - v) \, \partial_{x_2} V) \mathrm{d}x \\ &= \int_A (u - v) \times (\partial_{x_2} u \, \partial_{x_1} V - \partial_{x_1} u \, \partial_{x_2} V) \mathrm{d}x \\ &\quad + \int_A (u - v) \times (\partial_{x_2} v \, \partial_{x_1} V - \partial_{x_1} v \, \partial_{x_2} V) \mathrm{d}x. \end{aligned}$$

Then the statement of the lemma follows by the Cauchy-Schwartz inequality. $\square$

The main consequence of properties a) and b) of the function $\mathrm{abdeg}(u)$ is

Proposition 11. $\mathrm{abdeg}(u)$ *is close to integers uniformly in $u \in E_\varepsilon^\Lambda$ when ε is sufficiently small, i.e.*

c) $\sup\limits_{u \in E_\varepsilon^\Lambda} \mathrm{dist}(\mathrm{abdeg}(u), \mathbb{Z}) \to 0$ *as* $\varepsilon \to 0$.

Before proving this fact note that Proposition 11 immediately implies Proposition 2 stated in the Introduction.

Proof of Proposition 11. According to (3.2) and Lemma 10 we have

$$\sup_{u \in E_\varepsilon^\Lambda} \mathrm{dist}(\mathrm{abdeg}(u), \mathbb{Z}) \leq \sup_{u \in E_\varepsilon^\Lambda} \inf_{v \in E_0^\Lambda} |\mathrm{abdeg}(u) - \mathrm{abdeg}(v)| \leq \frac{2}{\pi} ||V||_{C^1} \Lambda^{\frac{1}{2}} \delta_\varepsilon, \tag{3.3}$$

where δ_ε is the following (nonsymmetric) distance

$$\delta_\varepsilon := \sup_{u \in E_\varepsilon^\Lambda} \mathrm{dist}_{L^2(A)}(u, E_0^\Lambda) \tag{3.4}$$

between E_ε^Λ and

$$E_0^\Lambda = \left\{ u \in H^1(A, S^1); \; E_0(u) = \frac{1}{2} \int_A |\nabla u|^2 \mathrm{d}x \leq \Lambda \right\}.$$

Show now that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. In view of (3.3) this yields the desired result. Assume by contradiction that $\delta_{\varepsilon_k} \geq c > 0$ for a sequence $\varepsilon_k \to 0$. By virtue of the Sobolev embeddings the supremum in (3.4) is attained on E_ε^Λ, i.e. $\delta_\varepsilon = \mathrm{dist}_{L^2(A)}(u_\varepsilon, E_0^\Lambda)$, where $E_\varepsilon(u_\varepsilon) \leq \Lambda$. We can extract a subsequence of (u_{ε_k}), still denoted (u_{ε_k}), that converges to a map u weakly in $H^1(A, \mathbb{R}^2)$. Thanks to Sobolev embeddings $u_{\varepsilon_k} \to u$ strongly in $L^2(A)$ and we have $u \in H^1(A, S^1)$, since $\int_A (|u_\varepsilon|^2 - 1)^2 dx \leq 4\Lambda\varepsilon^2$. Besides $E_0(u) \leq \Lambda$, by the lower weak semicontinuity of the Dirichlet integral. Thus $u \in E_0^\Lambda$ and $\delta_{\varepsilon_k} \leq \|u - u_{\varepsilon_k}\|_{L^2(A)} \to 0$. $\square$

The following result illustrates the relation of $\mathrm{abdeg}(u)$, which is not necessarily an integer with the standard notion of degree on a curve. It provides a simple criterion whether the constraint $\mathrm{abdeg}(u) \in [d - 1/2, d + 1/2]$ in (1.7) is satisfied in a particular case when u is a solution of equation (1.1).

Lemma 12. *Let $\mathcal{L} = \{x \in A; V(x) = 1/2\}$ denote the $1/2$ level set of V, where V is the solution of (1.9). ($\mathcal{L}$ is a smooth curve enclosing ω.) Then if a solution u of the GL equation (1.1) satisfies $|u| \leq 1$ in A and $E_\varepsilon(u) \leq \Lambda$, then* (i) *$|u| \geq 1/2$ on $\mathcal{L}$ and* (ii) *we have*

$$\mathrm{abdeg}(u) \in [d - 1/2, d + 1/2] \iff \deg(\frac{u}{|u|}, \mathcal{L}) = d, \tag{3.5}$$

when $\varepsilon \leq \varepsilon_1$, where $\varepsilon_1 = \varepsilon_1(\Lambda) > 0$ does not depend on u.

Proof. Consider the domain

$$A_{\delta'} = \{x \in A;\ \delta' < V(x) < 1 - \delta'\}, \tag{3.6}$$

where $0 < \delta' < 1/2$. It follows from Lemma 8 that u satisfies

$$|u| \geq 1/2 \quad \text{on } A_{\delta'}, \tag{3.7}$$

when $\varepsilon < \varepsilon_1'$ ($\varepsilon_1' = \varepsilon_1'(\delta') > 0$). This proves (i). We can now write $u = \rho e^{i\psi}$ ($\rho = |u| > 1/2$) on $A_{\delta'}$. Find an extension of ψ onto the whole domain A. To this end we pass to a conformal image of A.

It is well known (see,e.g. [1]) that there is a conformal mapping $\mathcal{G}$ of A onto the annulus $\mathcal{O}$ with $R = \exp(\pi/\mathrm{cap}(A))$ and $1/R$ as the outer and inner radii, correspondingly. Moreover, $\mathcal{G}$ is explicitly given by $\mathcal{G} = \exp(\frac{2\pi}{\mathrm{cap}(A)}((V - 1/2 + i\Psi))$, where Ψ is a (multivalued) harmonic conjugate of V. $\mathcal{G}$ maps $A_{\delta'}$ onto the annulus

$\mathcal{G}(A_{\delta'}) \subset \mathcal{O}$ whose outer and inner radii are $R' = \exp(\frac{2\pi}{\mathrm{cap}(A)}(1/2 - \delta'))$ and $1/R'$, correspondingly.

Now consider $\hat{\psi}(x) = \psi(G^{-1}(x))$ on $\mathcal{G}(A_{\delta'})$. We can extend $\hat{\psi}$ to the whole $\mathcal{O}$ by reflections $\hat{\psi}(x) := \hat{\psi}(x(R')^2/|x|^2)$ when $|x| \geq R'$ and $\hat{\psi}(x) := \hat{\psi}(x/(R'|x|)^2)$ when $|x| \leq 1/R'$, so that

$$\int_{\mathcal{O}} |\nabla\hat{\psi}|^2 \mathrm{d}x \leq \int_{\mathcal{G}(A_{\delta'})} |\nabla\hat{\psi}|^2 \mathrm{d}x + \int_{\mathcal{O}\setminus\mathcal{G}(A_{\delta'})} |\nabla\hat{\psi}|^2 \mathrm{d}x \leq 2\int_{\mathcal{G}(A_{\delta'})} |\nabla\hat{\psi}|^2 \mathrm{d}x, \tag{3.8}$$

when $0 < \delta' < 1/4$. Then (3.8), the conformal invariance of the Dirichlet integral and (3.7) imply

$$\begin{aligned}\int_{\mathcal{O}} |\nabla\hat{\psi}|^2 \mathrm{d}x \leq 2\int_{\mathcal{G}(A_{\delta'})} |\nabla\hat{\psi}|^2 \mathrm{d}x = 2\int_{A_{\delta'}} |\nabla\psi|^2 \mathrm{d}x \\ \leq 8\int_{A_{\delta'}} \rho^2|\nabla\psi|^2 \mathrm{d}x \leq 16E_\varepsilon(u) \leq 16\Lambda.\end{aligned} \tag{3.9}$$

The desired extension of ψ onto A is now given by $\tilde{\psi}(x) = \hat{\psi}(\mathcal{G}(x))$. Using again the conformal invariance of the Dirichlet integral, we see by (3.9) that $E_\varepsilon(\mathrm{e}^{i\tilde{\psi}}) = \frac{1}{2}\int_A |\nabla\tilde{\psi}|^2 \mathrm{d}x = \frac{1}{2}\int_{\mathcal{O}} |\nabla\hat{\psi}|^2 \mathrm{d}x \leq 8\Lambda$. Besides, since $\tilde{\psi} = \psi$ on $A_{\delta'}$ and $\rho = |u| \leq 1$, we get

$$\begin{aligned}\|u - \mathrm{e}^{i\tilde{\psi}}\|^2_{L^2(A)} = \int_{A\setminus A_{\delta'}} |u - \mathrm{e}^{i\tilde{\psi}}|^2 \mathrm{d}x + \int_{A_{\delta'}} (\rho - 1)^2 \mathrm{d}x \\ \leq 4|A \setminus A_{\delta'}| + \int_{A_{\delta'}} (\rho^2 - 1)^2 \mathrm{d}x \leq 4(|A \setminus A_{\delta'}| + \Lambda\varepsilon^2).\end{aligned} \tag{3.10}$$

Then, by choosing small δ' and $\varepsilon_1 = \varepsilon_1(\delta') > 0(\ \varepsilon_1(\delta') < \varepsilon_1')$, in view of Lemma 10, bounds $E_\varepsilon(\mathrm{e}^{i\tilde{\psi}}) \leq 8\Lambda$, $E_\varepsilon(u) \leq \Lambda$ and (3.10) we have $|\mathrm{abdeg}(u) - \mathrm{abdeg}(\mathrm{e}^{i\tilde{\psi}})| < 1/2$, when $\varepsilon < \varepsilon_1$. But $\mathrm{abdeg}(\mathrm{e}^{i\tilde{\psi}}) = \deg(\mathrm{e}^{i\tilde{\psi}}, \partial\Omega) = \deg(\mathrm{e}^{i\tilde{\psi}}, \mathcal{L}) = \deg(\frac{u}{|u|}, \mathcal{L})$, due to (3.2). Therefore if $\deg(\frac{u}{|u|}, \mathcal{L}) = d$ then $u \in E_\varepsilon^{\Lambda,d}$, and vice versa. Thus (ii) is proved. $\square$

4 Minimization among S^1-valued maps. Some upper and lower bounds for problem (1.7)

Consider the minimization problem

$$I_0(d, A') := \inf\{E_0(u);\ u \in H^1(A; S^1),\ \deg(u, \partial\omega') = \deg(u, \partial\Omega') = d\}, \tag{4.1}$$

where $E_0(u) = \int_{A'} |\nabla u|^2 \mathrm{d}x$, $A' = \Omega' \setminus \overline{\omega'}$, and ω', Ω' are smooth bounded simply connected domains in $\mathbb{R}^2$, such that $\overline{\omega'} \subset \Omega'$. This problem is a particular case of the minimization problem considered in [14](Chapter I).

Proposition 13. *[14] There exists a unique (up to multiplication on constants with unit modulus) solution u of the minimization problem (4.1), and u is a regular harmonic map in A' (i.e. $-\Delta u = u|\nabla u|^2$ in A') satisfying $u \times \frac{\partial u}{\partial \nu} = 0$ on $\partial A'$.*

When $A' = A$, then any minimizer u of (4.1) belongs to $\mathcal{J}_{dd}$. By (3.2) we also have $\mathrm{abdeg}(u) = d$. This yields the following (optimal) bound for (1.7), in the case when $p = q = d$.

Lemma 14. *We have $m_\varepsilon(d, d, d) \leq I_0(d, A)$ for any $\varepsilon > 0$.*

It is shown in [14] (Chapter I) that $I_0(d, A)$ can be expressed by

$$I_0(d, A) = \frac{1}{2} \int_A |\nabla h_0|^2 \mathrm{d}x, \tag{4.2}$$

where h_0 is the unique solution of the linear problem

$$\begin{cases} \Delta h_0 = 0 \quad \text{in } A \\ h_0 = 1 \quad \text{on } \partial\Omega, \quad h_0 = \text{Const} \quad \text{on } \partial\omega \\ \displaystyle\int_{\partial\Omega} \frac{\partial h_0}{\partial \nu} \mathrm{d}\sigma = 2\pi d. \end{cases} \tag{4.3}$$

h_0 and the solution V of (1.9) are related via $h_0 = 1 + 2\pi d(V - 1)/\mathrm{cap}(A)$, where $\mathrm{cap}(A)$ stands for H^1-capacity of A (see, e.g., [25]). Thus $I_0(d, A) = 2(\pi d)^2/\mathrm{cap}(A)$, and this clearly holds for any doubly connected domain A' in place of A. Therefore we have

Lemma 15. *$I_0(d, A')$ depends continuously on $\mathrm{cap}(A')$.*

By using this simple result, we obtain the following lower bound for the GL energy of solutions $u \in \mathcal{J}$ of the equation (1.1).

Lemma 16. *There is $\varepsilon_2 > 0$ such that for any solution $u \in \mathcal{J}_{lm}$ of GL equation (1.1) satisfying $E_\varepsilon(u) \leq \Lambda$, abdeg$(u) \in (d - 1/2, d + 1/2)$ we have*

$$E_\varepsilon(u) \geq I_0(d, A) - \frac{\pi}{2} + \pi(|d - l| + |d - m|), \tag{4.4}$$

when $\varepsilon < \varepsilon_2$, and ε_2 depends only on Λ.

Proof. Due to the maximum principle $|u| \leq 1$ on A. As in Lemma 12 we consider the domain $A_{\delta'}$ that is defined by (3.6) and depends on a positive parameter $\delta' < 1/2$ to be chosen later. Since $|u| \leq 1$ on A we can apply Lemma 8 to get the bound

$$|u| \geq 1 - \varepsilon \quad \text{in } A_{\delta'}, \tag{4.5}$$

for $\varepsilon < \varepsilon_2'$ ($\varepsilon_2' = \varepsilon_2'(\delta', \Lambda) > 0$). Consider now the map

$$\tilde{u} = \frac{1}{1-\varepsilon} \begin{cases} u & \text{if } |u| < 1 - \varepsilon \\ (1-\varepsilon)\frac{u}{|u|} & \text{otherwise.} \end{cases} \tag{4.6}$$

By (4.5) we have $|\tilde{u}| = 1$ on $A_{\delta'}$ and, according to Lemma 12, $\deg(\tilde{u}, \mathcal{L}) = d$ when $\varepsilon < \min\{\varepsilon_1, \varepsilon_2'\}$. Consequently, the degree of $\tilde{u}$ on both connected components of $\partial A_\delta'$ equals d, so that $\frac{1}{2}\int_{A_{\delta'}} |\nabla \tilde{u}|^2 \mathrm{d}x \geq I_0(d, A_{\delta'})$ (cf. (4.1)). Therefore, by using the obvious pointwise inequality $|\nabla \tilde{u}|^2 \geq 2|\partial_{x_1}\tilde{u} \times \partial_{x_2} u|$ and the integration by parts we get

$$\begin{aligned} \frac{1}{2}\int_A |\nabla \tilde{u}|^2 \mathrm{d}x \geq \frac{1}{2}\int_{A_{\delta'}} |\nabla \tilde{u}|^2 \mathrm{d}x \\ + \sum_{k=1,2} \Big| \int_{A_{\delta'}^{(k)}} \partial_{x_1}\tilde{u} \times \partial_{x_2}\tilde{u} \, \mathrm{d}x \Big| \geq I_0(d, A_{\delta'}) + \pi(|d-l| + |d-m|), \end{aligned} \tag{4.7}$$

where $A_{\delta'}^{(k)}$ and $k = 1, 2$ are respectively the outer and the inner connected components of $A \setminus A_{\delta'}$. On the other hand, it follows from (4.6) that $|u| \leq |\tilde{u}| \leq 1$. Therefore,

$$E_\varepsilon(\tilde{u}) \leq \frac{1}{(1-\varepsilon)^2} E_\varepsilon(u). \tag{4.8}$$

Bounds (4.7) and (4.8) yield (4.4) when δ' is such that $I_0(d, A_{\delta'}) \geq I_0(d, A) - \pi/4$ (cf. Lemma 15) and ε is sufficiently small. $\square$

5 From a vortexless minimizer to one with a single vortex

The main Theorem 3 is proved by induction on the "number of vortices" in minimizers. More precisely, given integer $d > 0$, we show the existence of minimizers of (1.7) for $p = q = d$, then pass to $p = d - 1$, $q = d$ and $p = d$, $q = d - 1$, e.t.c. The

key point of the proof is the induction step, when the degree changes by one on $\partial\omega$ or $\partial\Omega$. This change results in the rise of an additional vortex in a minimizer. For arbitrary p and q, satisfying the conditions of Theorem 3, this step is quite technical. This is why we consider here a particular case of transition from $p = q = d$ (no vortices) to $p = d$, $q = d - 1$ (one vortex). The transition from $p = q = d$ to $p = d - 1$ and $q = d$ is quite similar. We first establish

Lemma 17. *Given an integer $d > 0$. For sufficiently small ε, $\varepsilon \leq \varepsilon_3$ with $\varepsilon_3 > 0$, the infimum $m_\varepsilon(d, d, d)$ in (1.7) is always attained, and $m_\varepsilon(d, d, d) \leq I_0(d, A)$.*

Proof. Let u be a weak H^1-limit of a minimizing sequence $(u^{(k)})$. Since any minimizer v of problem (1.14) is an admissible testing map for problem (1.7), such a minimizing sequence exists and by using Price Lemma we obtain

$$E_\varepsilon(u) + \pi(|l - d| + |m - d|) \leq \liminf_{k\to\infty} E_\varepsilon(u^{(k)}) \leq \frac{1}{2}\int_A |\nabla v|^2 dx = I_0(d, A), \quad (5.1)$$

where $l = \deg(u, \partial\omega))$, $m = \deg(u, \partial\Omega)$. Due to Proposition 2 we have $\mathrm{abdeg}(u) \in (d - 1/2, d + 1/2)$ when $\varepsilon < \varepsilon_0$, therefore the first variation of (1.2) at u vanishes, i.e. u is a solution of equation (1.1). Indeed, thanks to Lemma 10, we have for any $w \in H^1_0(A; \mathbb{R}^2)$ with sufficiently small H^1-norm, $u^{(k)} + w$ is an admissible testing map when k is large, hence $E_\varepsilon(u+w) - E_\varepsilon(u) = \lim_{k\to\infty}(E_\varepsilon(u^{(k)}+w) - E_\varepsilon(u^{(k)}) \geq 0$ (where $\lim_{k\to\infty}$ denotes any partial limit), and we are done. Now, since u is a solution of (1.1), we can apply Lemma 16. That is we substitute (4.4) in (5.1), this leads to

$$|d - l| + |d - m| \leq \frac{1}{4}, \quad (5.2)$$

when $\varepsilon < \min\{\varepsilon_0, \varepsilon_2\}$, i.e. $l = m = d$ (since l, m and d are integers). Thus the infimum in (1.7) for $p = q = d$ is always attained when ε is sufficiently small. Lemma is proved. □

Next, we perform the transition from the minimization problem (1.7) for $p = q = d$ to that for $p = d$, $q = d - 1$ and show that $m_\varepsilon(d, d-1, d)$ is always attained when ε is sufficiently small. This is done by a comparison argument of $m_\varepsilon(d, d-1, d)$ with the energy $E_\varepsilon(u)$ of a minimizer u of (1.7) for $p = q = d$. We first describe the properties of such a minimizer.

In Section 5, it is shown that for small ε any minimizer u of (1.7) for $p = q = d$ is vortexless (see Remark 22), i.e. $u = \rho e^{i\psi}$ with smooth $\rho > 0$ and $\psi : A \to \mathbb{R} \setminus 2\pi d\mathbb{Z}$

(torus). It follows that we can write $u = \rho e^{id\theta}$, the $e^{i\theta}$ and $\nabla\theta$ being smooth maps globally on A. Then the boundary value problem (1.1)-(1.3) rewritten in terms of ρ and θ is

$$\begin{cases} \operatorname{div}(\rho^2\nabla\theta) = 0 \quad \text{in } A \\ \dfrac{\partial\theta}{\partial\nu} = 0 \quad \text{on } \partial A, \end{cases} \tag{5.3}$$

$$\begin{cases} -\Delta\rho + d^2|\nabla\theta|^2\rho + \dfrac{1}{\varepsilon^2}\rho(\rho^2-1) = 0 \quad \text{in } A \\ \rho = 1 \quad \text{on } \partial A. \end{cases} \tag{5.4}$$

We also have, in view of (2.2),

$$\nabla h = -d\rho^2\nabla^{\perp}\theta \quad \text{in } A, \tag{5.5}$$

where h is the solution of (2.1). It follows that (h,θ) defines orthogonal local coordinates in a neighborhood of $\partial\Omega$, thus straightening out the boundary. Really, it is straightforward to verify that

$$1 - h(\partial\omega) = \frac{1}{\operatorname{cap}(A)}\int_A \nabla h\cdot\nabla V\,\mathrm{d}x = 2\pi\operatorname{abdeg}(u)/\operatorname{cap}(A),$$

while abdeg(u) $\geq (d-1/2) > 0$. Then by applying the maximum principle to (2.4) we get, $1 > h(x) > h(\partial\omega)$ in A. This, in turn, implies, by Hopf's boundary lemma, that $\frac{\partial h}{\partial\nu} > 0$ on $\partial\Omega$; i.e. the map $(h,\theta) : A \to \mathbb{R}\times\mathbb{R}\setminus 2\pi\mathbb{Z}$ can be extended to a C^1-diffeomorphism of a one sided neighborhood of $\partial\Omega$ onto its image. Thus, there are some $\delta > 0$ and a domain $G_\delta \subset A$ such that

$$x \in G_\delta \to (h,\theta) \in \Pi_\delta = (1-\delta,1)\times\mathbb{R}\setminus 2\pi\mathbb{Z}$$

is a one-to-one correspondence, which extends to a C^1-diffeomorphsm of $\overline{G}_\delta$ onto $[1-\delta,1]\times\mathbb{R}\setminus 2\pi\mathbb{Z}$.

The following Proposition is crucial for existence of minimizers of (1.7) for $p = d$, $q = d-1$. In particular, combined with Lemma 14 it provides an independent of ε bound for $m_\varepsilon(d,d-1,d)$.

Proposition 18. *Let $u = \rho e^{id\theta}$, $\rho > 0$, be a minimizer of (1.7) for $p = q = d$. Assume that ε is so small that Proposition 2 holds with $\Lambda = I_0(d,A)$. Then there is a testing map $v \in \mathcal{J}_{d(d-1)}$ such that* abdeg$(u) \in (d-1/2, d+1/2)$ *and*

$$E_\varepsilon(v) - E_\varepsilon(u) < \pi. \tag{5.6}$$

In Section 7, the generalized version of Proposition 18 is used to show existence of minimizers with several vortices.

Proof of Proposition 18. We seek the testing map v in the form

$$v = \rho w_t, \tag{5.7}$$

with an unknown for the moment w_t. The following Lemma allows us to compare the energy $E_\varepsilon(u)$ of u with that of v.

Lemma 19. *If $w \in H^1(G', \mathbb{R}^2)$, $G' \subset A$, is such that $|w| = 1$ on G', then*

$$\int_{G'} (|\nabla(\rho w)|^2 + \frac{1}{2\varepsilon^2}(|\rho w|^2 - 1)^2) dx = \int_{G'} (|\nabla u|^2 + \frac{1}{2\varepsilon^2}(|u|^2 - 1)^2) dx + 2L_\varepsilon^{(d)}(w, G'),$$

where

$$L_\varepsilon^{(d)}(w, G') = \frac{1}{2}\int_{G'} \rho^2 |\nabla w|^2 dx - \frac{d^2}{2}\int_{G'} |\nabla\theta|^2 \rho^2 |w|^2 dx + \frac{1}{4\varepsilon^2}\int_{G'} \rho^4(|w|^2 - 1)^2 dx \tag{5.8}$$

This result is a variant of the factorization argument due to [16], its proof is presented in the end of this section.

Note that if $G' = G_\delta$ we can rewrite the functional (5.8) by using local coordinates (h, θ) as (cf. (5.5))

$$\begin{aligned} L_\varepsilon^{(d)}(w, G_\delta) = {} & \frac{d}{2}\int_{\Pi_\delta} |\partial_h w|^2 \rho^2 \mathrm{d}h\mathrm{d}\theta \\ & + \frac{1}{2d}\int_{\Pi_\delta} (|\partial_\theta w|^2 - d^2|w|^2)\mathrm{d}h\mathrm{d}\theta + \frac{1}{4\varepsilon^2}\int_{\Pi_\delta} \rho^2(|w|^2 - 1)^2 \frac{\mathrm{d}h\mathrm{d}\theta}{d|\nabla\theta|^2}. \end{aligned} \tag{5.9}$$

Instead of dealing with $L_\varepsilon^{(d)}(w, G_\delta)$ we will make use of the simplified functional with a quadratic penalty term,

$$M_\lambda(w) = \frac{1}{2d}\int_{\Pi_\delta} (d^2|\partial_h w|^2 + |\partial_\theta w|^2)\mathrm{d}h\mathrm{d}\theta + \frac{1}{2d}\int_{\Pi_\delta} (\lambda|w - \mathrm{e}^{i\theta}|^2 - d^2|w|^2)\mathrm{d}h\mathrm{d}\theta. \tag{5.10}$$

This last functional admits the separation of variables.

Now consider the map w_t that is given by $w_t = \mathrm{e}^{id\theta}$ in $A \setminus G_\delta$, and continued to G_δ as a minimizer of the functional $M_\lambda(w)$, where $\lambda \geq 2d^2$, with the following prescribed boundary data:

$$w_t = \mathrm{e}^{id\theta}\mathcal{F}_t(\mathrm{e}^{i\theta}) \quad \text{on } \partial\Omega, \tag{5.11}$$

$$w_t = \mathrm{e}^{id\theta} \quad \text{on } \partial G_\delta \setminus \partial\Omega, \tag{5.12}$$

where $\mathcal{F}_t(z) := \mathcal{C}_t(\bar{z})$ (bar stands for the complex conjugate), $\mathcal{C}_t(z) = \frac{z-(1-t)}{z(1-t)-1}$ is the classical Möbius conformal map from the unit disk onto itself, $t < 1$ is a positive parameter. Both parameters λ and t will be determined later. Since $\deg(\mathcal{F}_t, S^1) = -1$ and $\deg(\mathrm{e}^{i\theta}, \partial\Omega) = 1$, the standard properties of the topological degree implies that if v is as in (5.7), then

$$\deg(v, \partial\Omega) = d - 1, \ \deg(v, \partial\omega) = d. \tag{5.13}$$

The map w_t is well defined now, because the functional $M_\lambda(w)$ with Dirichlet condition on the boundary has a unique minimizer for $\lambda \geq 2d^2$. Moreover $|w_t| \leq 2$, since for if not then by taking $\tilde{w}_t = \frac{w_t}{|w_t|}\min\{|w_t|, 2\}$ in place of w_t the first term in (5.10) does not increase while the second one decreases, i.e. $M_\lambda(\tilde{w}_t) < M_\lambda(w_t)$, it is a contradiction.

Note that under the following choice of λ,

$$\lambda := \max\left\{\frac{9}{2\varepsilon^2 \inf_{G_\delta} |\nabla\theta|^2}, \, 2d^2\right\}$$

($|\nabla\theta| > 0$ on the closure of G_δ) we have

$$\begin{aligned}\rho^2(|w_t|^2 - 1)^2 \leq (|w_t| - 1)^2(|w_t| + 1)^2 \leq |w_t - \mathrm{e}^{id\theta}|^2(|w_t| + 1)^2 \\ \leq 9|w_t - \mathrm{e}^{id\theta}|^2 \leq 2\varepsilon^2\lambda|\nabla\theta|^2|w_t - \mathrm{e}^{id\theta}|^2 \quad \text{in } G_\delta,\end{aligned}$$

thanks to the bounds $|w_t| \leq 2$ and $\rho \leq 1$. It follows that $L_\varepsilon^{(d)}(w_t, G_\delta) \leq M_\lambda(w_t)$. We get now, by virtue of Lemma 19, that $v = \rho w_t$ satisfies

$$E_\varepsilon(v) \leq E_\varepsilon(u) + M_\lambda(w_t). \tag{5.14}$$

We can obtain a representation for $M_\lambda(w_t)$ in separated variables. Namely, expanding $z^d\mathcal{F}_t(z)$ on S^1 as

$$z^d\mathcal{F}_t(z) = (1-t)z^d + t(t-2)\sum_{k=0}^{\infty}(1-t)^k z^{d-k-1},$$

we have

$$w_t = (1 - tf_{-1}(h))\mathrm{e}^{id\theta} + t(t-2)\sum_{k=0}^{\infty}(1-t)^k f_k(h)\mathrm{e}^{-i(k-d+1)\theta}, \tag{5.15}$$

where $f_k(h)$ satisfy, according to (5.11, 5.12),

$$f_k(1-\delta) = 0, \quad f_k(1) = 1. \tag{5.16}$$

Substitute (5.15) into (5.10) to obtain

$$M_\lambda(w_t) = \frac{t^2\pi}{d}\Phi_{-1}(f_{-1}) + \frac{t^2\pi}{d}\sum_{k=0}^{\infty}(t-2)^2(1-t)^{2k}\Phi_k(f_k), \tag{5.17}$$

where

$$\Phi_k(f_k) = \int_{1-\delta}^{1}\left(d^2|f_k'(h)|^2 + ((k-d+1)^2 + \lambda - d^2)|f_k(h)|^2\right)\mathrm{d}h. \tag{5.18}$$

Minimizing (5.18) under the conditions (5.16) we get

$$f_k(h) = \frac{\mathrm{e}^{k_+(h-1)}}{1-\mathrm{e}^{(k_- - k_+)\delta}} + \frac{\mathrm{e}^{k_-(h-1)}}{1-\mathrm{e}^{(k_+ - k_-)\delta}}, \tag{5.19}$$

where $k_\pm = \pm\frac{1}{d}\sqrt{(k-d+1)^2+\lambda-d^2}$. Therefore, we have

$$\Phi_k(f_k) = d(k-d+1)(1+\frac{\lambda-d^2}{2k^2} + O(1/k^3)), \text{ as } k\to\infty. \tag{5.20}$$

Finally, using (5.20) in (5.17), we obtain

$$\begin{aligned} M_\lambda(w_t) &\le \pi((1-t)^2-1)^2\sum_{k=0}^{\infty}k(1-t)^{2k} + 2\pi t^2(\lambda-d^2)\sum_{k=1}^{\infty}\frac{(1-t)^{2k}}{k} + Ct^2 \\ &= \pi(1-2t-2t^2(\lambda-d^2)\log(1-(1-t)^2)) + (C+\pi)t^2. \end{aligned} \tag{5.21}$$

Observe that the right hand side of (5.21) is strictly less than π when $t>0$ is chosen sufficiently small. By (5.14), for such t the map $v=\rho w_t$ satisfies (5.6).

It remains only to show that $\mathrm{abdeg}(v) \in (d-1/2, d+1/2)$. To this end note that by (5.15) and (5.19) $w_t \to \mathrm{e}^{id\theta}$ pointwise in G_δ as $t\to 0$. Therefore $\rho w_t \to \rho\mathrm{e}^{id\theta}(=u)$ weakly in $H^1(A)$, so that $\mathrm{abdeg}(\rho w_t)\to\mathrm{abdeg}(u)$. On the other hand, by Proposition 2, we know $d-1/2 < \mathrm{abdeg}(u) < d+1/2$. Thus, after possibly passing to a smaller t, $v=\rho w_t$ satisfies the required property. □

Now we have that under the conditions of Proposition 18, there exists a minimizing sequence $(u^{(k)})$ of admissible testing maps in problem (1.7) for $p=d$, $q=d-1$ such that $\lim_{k\to\infty}E_\varepsilon(u^{(k)}) < m_\varepsilon(d,d,d)+\pi$ and $u^{(k)}$ weakly H^1-converge to a map

$u \in \mathcal{J}$. Moreover, any minimizing sequence has a subsequence with the same properties. Show that any weak limit u is also an admissible map. Let $l = \deg(u, \partial\omega)$, $m = \deg(u, \partial\Omega)$. By virtue of Lemma 16 (in the same way us in Lemma 17 one shows that u satisfies (1.1)) and Lemma 9 we have

$$I_0(d, A) - \frac{\pi}{2} + \pi(2|d-l| + |d-1-m| + |d-m|)$$
$$\leq E_\varepsilon(u) + \pi(|d-l| + |d-1-m|) < m_\varepsilon(d, d, d) + \pi, \quad (5.22)$$

since $\operatorname{abdeg}(u) = \lim_{k\to\infty} \operatorname{abdeg}(u^{(k)}) \in [d - 1/2, d + 1/2]$. Due to Lemma 17 $m_\varepsilon(d, d, d) \leq I_0(d, A)$ so that (5.22) implies that $l = d$ and either $m = d - 1$ or $m = d$. In the last case, u becomes an admissible map in problem (1.7) for $p = q = d$ and therefore $E_\varepsilon(u) \geq m_\varepsilon(d, d, d)$, which contradicts the last inequality in (5.22). Thus, $u \in \mathcal{J}_{d(d-1)}$, $\operatorname{abdeg}(u) \in [d - 1/2, d + 1/2]$, i.e. u is in the set of admissible testing maps of problem (1.7) for $p = d$, $q = d - 1$.

Proof of Lemma 19. We have, by using (5.4),

$$\begin{aligned}\int_{G_\delta} |\nabla(\rho w)|^2 \mathrm{d}x &= \int_{G_\delta} (\rho^2|\nabla w|^2 + \nabla\rho \cdot \nabla(\rho(|w|^2 - 1)) + |\nabla\rho|^2)\mathrm{d}x \\ &= \int_{G_\delta} (\rho^2|\nabla w|^2 + d^2\rho^2|\nabla\theta|^2 + \frac{1}{\varepsilon^2}\rho^2(\rho^2 - 1))\mathrm{d}x \\ &\quad - \int_{G_\delta} (d^2\rho^2|\nabla\theta|^2|w|^2 + \frac{1}{\varepsilon^2}\rho^2(\rho^2 - 1)|w|^2 - |\nabla\rho|^2)\mathrm{d}x.\end{aligned}$$

Then simple algebraic manipulations give the required result. $\square$

6 Asymptotic behavior of local minimizers

In the previous section, we established the existence of minimizers of (1.2) in $\mathcal{J}_{dd}^{(d)}$ and demonstrated the first induction step of the proof of Theorem 3 that consists in transition from $p = q = d$ to $p = d$, $q = d - 1$ in (1.7). (In fact, modulo the assumption that minimizers in $\mathcal{J}_{dd}^{(d)}$ are vortexless, we actually proved the existence of minimizers in $\mathcal{J}_{(d-1)d}^{(d)}$ and $\mathcal{J}_{d(d-1)}^{(d)}$.) In order to show the induction step for any integer $p \leq d$ and $q \leq d$, we need to establish some properties of minimizers of (1.7), and we are especially interested in their behavior near the boundary. At this point, we assume we are given a family $\{u_\varepsilon\}$ of minimizers for (1.7) and

$$E_\varepsilon(u_\varepsilon) \leq \Lambda := I_0(d, A) + \pi(|d - p| + |d - q|). \quad (6.1)$$

Also, we suppose also $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(\Lambda) > 0$ is as in Proposition 2. It follows that maps u_ε are local minimizers of $E_\varepsilon(u)$ in $\mathcal{J}$ and therefore they satisfy (1.1), (1.3).

We will use the notations: $B_\varepsilon(y) = \{x \in \mathbb{R}^2 : |x - y| < \varepsilon\}$, $\rho_\varepsilon(x) = |u_\varepsilon(x)|$, $h_\varepsilon(x)$ is the unique solution of (2.1) (associated to u_ε), and $\mathcal{L}$ is the contour as in Lemma 12. The contour $\mathcal{L}$ separates the two open subdomains $Q^\pm$ in A, where Q^+ is the domain enclosed by $\partial\Omega$ and $\mathcal{L}$ and $Q^- = A \setminus (Q^+ \cup \mathcal{L})$. We also set $Q^\pm_\varepsilon = \{x \in Q^\pm;\, \rho^2_\varepsilon(x) \leq 1 - \varepsilon^{1/2}\}$.

6.1 Proof of Theorem 4

Since $|\nabla h_\varepsilon| \leq |\nabla u_\varepsilon|$ (by Lemma 6), the family $\{h_\varepsilon\}$ is bounded in $H^1(A)$, and therefore there is a sequence $\varepsilon_k \to 0$ such that

$$h_{\varepsilon_k} \to h \quad \text{weakly in } H^1(A), \text{ as } k \to \infty. \tag{6.2}$$

In order to identify h, we make use of Lemma 8 to obtain that, up to a subsequence, maps u_{ε_k} converge to a S^1-valued map u in $C^1_{\rm loc}(A)$. Since $\partial_{x_1} u \times \partial_{x_2} u = 0$ a.e. in A, we have $\Delta h_{\varepsilon_k} = 2\partial_{x_1} u_{\varepsilon_k} \times \partial_{x_2} u_{\varepsilon_k} \to 0$ in $C^0_{\rm loc}(A)$, thus h is a harmonic function. Moreover, $h = 1$ on $\partial\Omega$ and $h = \text{Const}$ on $\partial\omega$. On the other hand,

$$\operatorname{abdeg}(u_{\varepsilon_k}) = \frac{1}{2\pi}\int_A \nabla h_{\varepsilon_k} \cdot \nabla V \mathrm{d}x \to \frac{1}{2\pi}\int_A \nabla h \cdot \nabla V \mathrm{d}x = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial h}{\partial \nu}\mathrm{d}s.$$

According to the property c) of $\operatorname{abdeg}(\,\cdot\,)$ (see Proposition 11 in Section 3), $\operatorname{abdeg}(u_\varepsilon) \to d$, as $\varepsilon \to 0$, therefore $h = h_0$ (where h_0 is the unique solution of (4.3)) and the convergence in (6.2) holds for the whole family $\{h_\varepsilon\}$. Thus, applying again Lemma 8, we obtain

$$h_\varepsilon \to h_0 \quad \text{in } C^1_{\rm loc}(\mathrm{A}) \text{ and weakly in } H^1(A), \text{ as } \varepsilon \to 0. \tag{6.3}$$

By (6.1) and Lemma 8, maps u_ε converge, up to a subsequence, to $u \in H^1(A; S^1)$ in $C^1_{\rm loc}(A)$ and weakly in $H^1(A)$. Moreover, $\operatorname{abdeg}(u) = d$ and in view of (6.3) $|\nabla u| = |\nabla h_0|$ a.e. in A. It follows that u is a solution of the minimization problem (1.14).

In order to demonstrate the energy convergence stated in Theorem 4, we argue as follows: by using two pointwise equalities $|\nabla u_\varepsilon|^2 = 2\partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon + 4|\partial_{\bar z} u_\varepsilon|^2$ and

$|\nabla u_\varepsilon|^2 = -2\partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon + 4|\partial_z u_\varepsilon|^2$ and the pointwise inequality $|\nabla u_\varepsilon| \geq |\nabla h_\varepsilon|$, we have

$$\frac{1}{2}\int_A |\nabla u_\varepsilon|^2 \mathrm{d}x \geq -\int_{Q_\varepsilon^+} \partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon \mathrm{d}x + 2\int_{Q_\varepsilon^+} |\partial_z u_\varepsilon|^2 \mathrm{d}x$$
$$+ \int_{Q_\varepsilon^-} \partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon \mathrm{d}x + 2\int_{Q_\varepsilon^-} |\partial_{\bar z} u_\varepsilon|^2 \mathrm{d}x + \frac{1}{2}\int_{A\setminus(Q_\varepsilon^+ \cup Q_\varepsilon^-)} |\nabla h_\varepsilon|^2 \mathrm{d}x. \quad (6.4)$$

Let us estimate the right hand side of (6.4) from below. Introducing $\sigma_\varepsilon(x) = \max\{\rho_\varepsilon^2(x), 1-\varepsilon^{1/2}\}$, we have (by (2.3) (2.4))

$$\mathrm{div}\left(\frac{1}{\sigma_\varepsilon(x)}\nabla h_\varepsilon\right) = \frac{2}{1-\varepsilon^{1/2}} \begin{cases} 0 \quad \text{in} \quad A \setminus (Q_\varepsilon^+ \cup Q_\varepsilon^-); \\ \partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon \text{ otherwise.} \end{cases} \quad (6.5)$$

Integrating (6.5) over Q^+, we get for sufficiently small ε,

$$\frac{2}{1-\varepsilon^{1/2}} \int_{Q_\varepsilon^+} \partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon \mathrm{d}x = \int_{\partial\Omega} \frac{\partial h_\varepsilon}{\partial \nu} \mathrm{d}s - \int_{\mathcal{L}} \frac{\partial h_\varepsilon}{\partial \nu} \frac{\mathrm{d}s}{\rho_\varepsilon^2(x)}$$
$$= \int_{\partial\Omega} u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial \tau} \mathrm{d}s - \int_{\mathcal{L}} \frac{u_\varepsilon}{|u_\varepsilon|} \times \frac{\partial}{\partial \tau}\frac{u_\varepsilon}{|u_\varepsilon|} \mathrm{d}s = 2\pi(q-d),$$

where we have used Lemma 8 and Lemma 12. Thus, we have

$$\int_{Q_\varepsilon^+} \partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon \mathrm{d}x = (1-\varepsilon^{1/2})\pi(q-d). \quad (6.6)$$

Similarly, integrating (6.5) over Q^- we obtain

$$\int_{Q_\varepsilon^-} \partial_{x_1} u_\varepsilon \times \partial_{x_2} u_\varepsilon \mathrm{d}x = (1-\varepsilon^{1/2})\pi(d-p). \quad (6.7)$$

In order to estimate the last term in the right hand side of (6.4), we write it as

$$\int_{A\setminus(Q_\varepsilon^+ \cup Q_\varepsilon^-)} |\nabla h_\varepsilon|^2 \mathrm{d}x = \int_{A\setminus(Q_\varepsilon^+ \cup Q_\varepsilon^-)} |\nabla h_\varepsilon - \nabla h_0|^2 \mathrm{d}x$$
$$+ \int_A (2\nabla h_\varepsilon - \nabla h_0)\cdot \nabla h_0 \mathrm{d}x - \int_{Q_\varepsilon^+ \cup Q_\varepsilon^-} (2\nabla h_\varepsilon - \nabla h_0)\cdot \nabla h_0 \mathrm{d}x,$$

and note that by virtue of (6.1) the measure of $Q_\varepsilon^+ \cup Q_\varepsilon^-$ vanishes as $\varepsilon \to 0$, so that

$$\int_{A\setminus(Q_\varepsilon^+ \cup Q_\varepsilon^-)} |\nabla h_\varepsilon|^2 \mathrm{d}x = \int_{A\setminus(Q_\varepsilon^+ \cup Q_\varepsilon^-)} |\nabla h_\varepsilon - \nabla h_0|^2 \mathrm{d}x + \int_A |\nabla h_0|^2 \mathrm{d}x + o(1). \quad (6.8)$$

Thus (6.6-6.8, 6.4, 6.1) imply $E_\varepsilon(u_\varepsilon) \to E_0(u) + \pi(|d-p| + |d-q|)$. $\square$

As a byproduct of the above proof by, (6.6)-(6.8), (6.4) and (6.1) we get

$$\int_A (|u_\varepsilon|^2 - 1)^2 \mathrm{d}x = o(\varepsilon^2), \tag{6.9}$$

$$\int_{A\setminus(Q_\varepsilon^+ \cup Q_\varepsilon^-)} |\nabla h_\varepsilon - \nabla h_0|^2 \mathrm{d}x = o(1), \tag{6.10}$$

$$\int_{Q_\varepsilon^+} |\partial_z u_\varepsilon|^2 \mathrm{d}x = o(1) \quad \int_{Q_\varepsilon^-} |\partial_{\bar{z}} u_\varepsilon|^2 \mathrm{d}x = o(1). \tag{6.11}$$

6.2 Properties of minimizers of (1.7) for small ε

First, by using (6.9) and the following methods of [14] we get that ρ_ε converges to 1 uniformly on compacts in A. Moreover, we have

Lemma 20. *For any $\mu > 0$ we have*

$$\sup\{\mathrm{dist}(y, \partial A);\ y \in A, \rho_\varepsilon^2(y) < 1 - \mu\} = o(\varepsilon). \tag{6.12}$$

Proof. Assume by contradiction, that for a sequence $\varepsilon_k \to 0$ and $\gamma > 0$ we have $\rho_{\varepsilon_k}^2(y_k) < 1 - \mu$ and $\mathrm{dist}(y_k, \partial A) \geq \gamma\varepsilon_k$. Due to (2.6), $|\nabla|u_{\varepsilon_k}|^2| \leq \alpha/\varepsilon_k$ in $B_{\lambda\varepsilon_k}(y_k)$, where $0 < \lambda < \gamma$ and $\alpha (= \alpha(\lambda))$ is independent of ε_k. It follows that $|u_{\varepsilon_k}(x)|^2 < 1 - \mu + \delta\alpha$ when $x \in B_{\delta\varepsilon_k}(y_k)$ and $\delta < \lambda$. Then $B_{\delta\varepsilon_k}(y_k) \subset A$ and

$$\frac{1}{\varepsilon_k^2} \int_{B_{\delta\varepsilon_k}(y_k)} (|u_{\varepsilon_k}|^2 - 1)^2 \mathrm{d}x \geq \pi(\mu - \delta\alpha)^2\delta^2 > 0,$$

as soon as $0 < \delta < \min\{\lambda, \mu/(2\alpha)\}$. This contradicts (6.9). $\square$

Important properties of u_ε and h_ε, in a vicinity of the boudary ∂A, are established in

Lemma 21. *For any $0 < \mu < 1$ and $\kappa < 1$ there are $\hat{\varepsilon}_1(\mu), \hat{\varepsilon}_2(\mu, \kappa) > 0$ such that if $\rho_\varepsilon^2(y) \leq 1 - \mu$ then*

(a) *for $\varepsilon < \hat{\varepsilon}_1(\mu)$ we have $h_\varepsilon(y) \geq 1 + \mu/4$ if $\mathrm{dist}(y, \partial\Omega) < \varepsilon$ and $h_\varepsilon(y) \leq r_\varepsilon(\partial\omega) - \mu/4$ if $\mathrm{dist}(y, \partial\omega) < \varepsilon$;*

(b) *for $\varepsilon < \hat{\varepsilon}_2(\mu, \kappa)$ we have*

$$\frac{1}{2} \int_{A \cap B_\varepsilon(y)} |\nabla u_\varepsilon|^2 \mathrm{d}x \geq \kappa\pi. \tag{6.13}$$

Proof.(by contradiction) Let us assume that either (a) or (b) is violated for a sequence $\varepsilon_k \to 0$ and some $y = y_k$ such that $h_{\varepsilon_k}(y_k) \leq 1-\mu$. According to Lemma 20, $y_k \to \partial A$. For the definiteness we suppose that $y_k \to \partial\Omega$, then (by Lemma 20)

$$\operatorname{dist}(y_k, \partial\Omega) = o(\varepsilon_k). \tag{6.14}$$

Let u_ε be continued in ω in such a way that $\|u_\varepsilon\|_{H^1(\Omega)} \leq C\|u_\varepsilon\|_{H^1(A)}$ and $|u_\varepsilon| \leq 1$ in Ω, where C is independed of ε. We also assume that $h_\varepsilon = h_\varepsilon(\partial\omega)$ on ω. Following [12] we rescale u_{ε_k} and h_{ε_k} by a conformal map that 'moves' y_k away from the boundary. Fix a conformal mapping η from Ω onto the unit disk $B_1(0)$. We introduce the conformal map $\zeta_k(z) = (z-\eta(y_k))/(\bar{\eta}(y_k)z-1)$ from $B_1(0)$ onto itself and set $U_k(z) = u_{\varepsilon_k}(\eta^{-1}(\zeta_k(z)))$, $H_k(z) = h_{\varepsilon_k}(\eta^{-1}(\zeta_k(z)))$. It is easy to see that $\|U_k\|_{H^1(\Omega)} \leq C$ and $\|H_k\|_{H^1(\Omega)} \leq C$ with some C independent of k. Therefore, without loss of generality, we can assume that U_k and H_k H^1-weakly converge to limits U and H, respectively, as $k \to \infty$.

Arguing, as in [12](Section 4), we can show that $U_k \to U$ in $C^1_{\rm loc}(B_1(0))$ and that $\Delta U = 0$ in $B_1(0)$. Therefore, $|U(0)|^2 = \lim_{k\to\infty}|U_k(0)|^2 = \lim_{k\to\infty}|u_{\varepsilon_k}(y_k)|^2 \leq 1-\mu$. We also have $|U| = 1$ a.e. on S^1. Show now that $\partial_z U = 0$ in $B_1(0)$. Indeed, by the maximum principle $|U| < 1$ in $B_1(0)$, hence $\max_{B_t(0)}|U_k(z)|^2 < 1-\varepsilon_k^{1/2}$ for any fixed $0 < t < 1$ and sufficiently large k. It follows that for such k we have $\eta^{-1}(\zeta_k(B_t(0))) \subset Q^+_\varepsilon$. Then in view of (6.11) we get, by using the conformality of the maps η^{-1} and ζ_k,

$$\int_{B_t(0)} |\partial_z U_k|^2 dx = \int_{\eta^{-1}(\zeta_k(B_t(0)))} |\partial_z u_{\varepsilon_k}|^2 dx \leq \int_{A^+_\varepsilon} |\partial_z u_{\varepsilon_k}|^2 dx \to 0$$

This implies that $\partial_z U = 0$ in $B_1(0)$.

In order to show (b) we use the pointwise equalities $\frac{1}{2}|\nabla U|^2 = -\partial_{x_1}U \times \partial_{x_2}U + \frac{1}{4}|\partial_z U|^2$ and $\partial_z U = 0$ to obtain

$$\frac{1}{2}\int_{B_1(0)} |\nabla U|^2 \mathrm{d}x = -\int_{B_1(0)} \partial_{x_1}U \times \partial_{x_2}U \mathrm{d}x = -\pi \deg(U, S^1).$$

As $U \not\equiv$ Const, we therefore have $\frac{1}{2}\int_{B_1(0)}|\nabla U|^2 \mathrm{d}x \geq \pi$. It follows that there is $0 < t < 1$ such that

$$\frac{1}{2}\int_{B_t(0)} |\nabla U|^2 dx > \kappa\pi. \tag{6.15}$$

The image $\zeta_k(B_t(0))$ of the disk $B_t(0)$ is the disk $B_{t_k}(\xi_k)$ with the radius $t_k = \frac{t(1-|\eta(y_k)|^2)}{1-t^2|\eta(y_k)|^2}$ and the center at $\xi_k = \frac{1-t^2}{1-t^2|\eta(y_k)|^2}$. According to (6.14) $t_k = o(\varepsilon_k)$ for

$k \to \infty$, hence $\eta^{-1}(B_{t_k}(\xi_k)) \subset B_{\varepsilon_k}(y_k)$ when k is sufficiently large. Then, by using the conformal invariance and lower semicontinuity of the Dirichlet integral, and bound (6.15), we get

$$\int_{B_{\varepsilon_k}(y_k)} |\nabla u_{\varepsilon_k}|^2 \mathrm{d}x \geq \int_{\eta^{-1}(\zeta_k(B_t(0)))} |\nabla u_{\varepsilon_k}|^2 \mathrm{d}x = \int_{B_t(0)} |\nabla U_k|^2 \mathrm{d}x > 2\kappa\pi \text{ as } k \to \infty.$$

In order to show that $h_{\varepsilon_k}(y_k) = H_k(0) > 1 + \mu/4$ when $k \to \infty$ we note that the system (2.1) is conformally invariant, i.e.

$$\nabla^{\perp} H_k = (U_k \times \partial_{x_1} U_k, U_k \times \partial_{x_2} U_k) \quad \text{in} \quad \zeta_k^{-1}(\eta(A)).$$

Then, bearing in mind the convergence properties of U_k, we obtain that $H_k \to H$ in $C^1_{\mathrm{loc}}(B_1(0))$ and

$$\nabla^{\perp} H = (U \times \partial_{x_1} U, U \times \partial_{x_2} U) = -\frac{1}{2}\nabla^{\perp}(|U|^2) \quad \text{in} \quad B_1(0),$$

where we have used the fact that $\partial_z U = 0$ in $B_1(0)$. Since $H = |U| = 1$ on $\partial B_1(0)$ we have $H = \frac{3}{2} - \frac{1}{2}|U|^2$ in $B_1(0)$, therefore

$$\lim_{k\to\infty} h_{\varepsilon_k}(y_k) = \lim_{k\to\infty} H_k(0) = \frac{3}{2} - \frac{1}{2}|U(0)|^2 \geq 1 + \frac{\mu}{2}.$$

Lemma is proved. □

Remark 22. Lemma 21 implies that in the case when $p = q = d$ minimizers of (1.7) are vortexless for sufficiently small ε. Really, by Theorem 4 they H^1-strongly converge, up to a subsequence, as $\varepsilon \to 0$ to a minimizing harmonic map $u \in \mathcal{J}^{(d)}_{dd}$. On the other hand (6.13), exhibits the energy concentration property near zeros of minimizers, which is incompatible with the strong H^1-convergence.

The following result, describing the structure of the function h_ε for small ε plays a crucial role in the proof of the main technical result (Lemma 25) in Section 7.

Lemma 23. *We have, for small ε, $\varepsilon < \varepsilon_4$ (where $\varepsilon_4 > 0$),*

(i) $\rho_\varepsilon^2(x) \geq 1/2$ *when* $h_\varepsilon(\partial\omega) - 1/8 \leq h_\varepsilon(x) \leq 9/8$ *and* $h_\varepsilon(\partial\omega) < \min_{\mathcal{L}} h_\varepsilon(x) \leq \max_{\mathcal{L}} h_\varepsilon(x) < h_\varepsilon(\partial\Omega)$, *while if* $\rho_\varepsilon^2(x) < 1/2$ *then either*

$$\mathrm{dist}(x, \partial\Omega) < \mathrm{dist}(\mathcal{L}, \partial\Omega) \text{ and } h_\varepsilon(x) > 9/8$$

or

$$\mathrm{dist}(x, \partial\omega) < \mathrm{dist}(\mathcal{L}, \partial\omega) \text{ and } h_\varepsilon(x) < h_\varepsilon(\partial\omega) - 1/8;$$

(ii) there are $x_\varepsilon^* \in \partial\Omega$, $x_\varepsilon^{**} \in \partial\omega$ such that $\frac{\partial h_\varepsilon}{\partial \nu}(x_\varepsilon^*) > 0$ and $\frac{\partial h_\varepsilon}{\partial \nu}(x_\varepsilon^{**}) > 0$.

Proof. (i) follows from Lemma 20, Lemma 21 and the convergence properties of h_ε as $\varepsilon \to 0$ established in the course of the proving Theorem 4. To prove (ii) we argue as follows. Let $\hat{\varepsilon}_2(\mu, \kappa)$ be the best (biggest) constant in Lemma 21. Then $\hat{\varepsilon}_2(\mu, \kappa)$ is increasing in μ and decreasing in κ. For $k = 1, 2, \ldots$ set

$$\hat{\mu}_\varepsilon = 1/k \quad \text{when} \quad \min\{\hat{\varepsilon}_2(\tfrac{1}{k+1}, \tfrac{k}{k+1}), 1/(k+1)\} \leq \varepsilon < \min\{\hat{\varepsilon}_2(\tfrac{1}{k}, \tfrac{k-1}{k}), 1/k\}.$$

Then $\hat{\mu}_\varepsilon \to 0$ as $\varepsilon \to 0$ and (6.13) is satisfied with $\kappa = 1 - \hat{\mu}_\varepsilon$ when $\rho_\varepsilon^2(y) < 1 - \hat{\mu}_\varepsilon$; the same being true when $\hat{\mu}_\varepsilon$ is replaced by $\mu_\varepsilon = \max\{\hat{\mu}_\varepsilon, \varepsilon^{1/2}\}$. We pick a point $x_\varepsilon^{(1)}$ in A such that $\rho_\varepsilon^2(x_\varepsilon^{(1)}) < 1 - \mu_\varepsilon$; then we pick a point $x_\varepsilon^{(2)}$ in $A \setminus B_{2\varepsilon}(x_\varepsilon^{(1)})$ such that $\rho_\varepsilon^2(x_\varepsilon^{(1)}) < \mu_\varepsilon$, e.t.c. unless for some K_ε we have $\rho_\varepsilon^2(x) \geq 1 - \mu_\varepsilon$ on $A \setminus \cup_{k=1}^{K_\varepsilon} B_{2\varepsilon}(x_\varepsilon^{(k)})$. By the construction of μ_ε, since disks $B_\varepsilon(x_\varepsilon^{(k)})$ are disjoint,

$$\frac{1}{2}\int_A |\nabla u^\varepsilon|^2 dx \geq \frac{1}{2}\sum_1^{K_\varepsilon} \int_{A \cap B_\varepsilon(x_\varepsilon^{(k)})} |\nabla u^\varepsilon|^2 dx \geq K_\varepsilon(1 - \mu_\varepsilon)\pi.$$

Therefore by (6.1) we have a uniform bound $K_\varepsilon \leq C$. Arguing as in [14] (Chapter IV, Theorem IV.1) we can increase the radii of disks to $\varepsilon\lambda > 2\varepsilon$ (with λ independent of ε) and take a subset I_ε of $\{1, \ldots, K_\varepsilon\}$ such that

$$\cup_{k \in I_\varepsilon} B_{\varepsilon\lambda}(x_\varepsilon^k) \supset \cup_{k=1}^{K_\varepsilon} B_{2\varepsilon}(x_\varepsilon^k) \quad \text{and} \quad \operatorname{dist}(x_\varepsilon^{k'}, x_\varepsilon^k) > 4\varepsilon\lambda \quad \text{for different} \quad k, k' \in I_\varepsilon.$$

We also have

$$\rho_\varepsilon^2(x) \geq 1 - \mu_\varepsilon \text{ on } A \setminus \cup_{k \in I_\varepsilon} B_{\varepsilon\lambda}(x_\varepsilon^k)$$

Assume h_ε and h_0 extend to the whole $\mathbb{R}^2$ and set $h_\varepsilon = h_\varepsilon(\partial\omega)$, $h_0 = h_0(\partial\omega)$ in ω, and $h_\varepsilon = h_0 = 1$ in $\mathbb{R}^2 \setminus \Omega$. Since $\rho_\varepsilon^2(x) \geq 1 - \mu_\varepsilon \geq 1 - \varepsilon^{1/2}$ in $D_\varepsilon^{(k)} = B_{2\lambda\varepsilon}(x_\varepsilon^{(k)}) \setminus B_{\lambda\varepsilon}(x_\varepsilon^{(k)})$, by (6.10) we have

$$\int_{D_\varepsilon^{(k)}} |\nabla h_\varepsilon|^2 \mathrm{d}x \leq 2\int_{D_\varepsilon^{(k)}} (|\nabla(h_\varepsilon - h_0)|^2 + |\nabla h_0|^2)\mathrm{d}x \to 0 \text{ as } \varepsilon \to 0.$$

Then, writing the integral over $D_\varepsilon^{(k)}$ in the polar coordinates with the center at $x_\varepsilon^{(k)}$ and using Fubini's theorem, we can find $\lambda_\varepsilon^{(k)}$, $\lambda\varepsilon \leq \lambda_\varepsilon^{(k)} \leq 2\lambda\varepsilon$, such that

$$\int_{|x - x_\varepsilon^{(k)}| = \lambda_\varepsilon^{(k)}} |\nabla h_\varepsilon|^2 d\sigma = o(1/\varepsilon).$$

Therefore, by the Cauchy-Shwartz inequality,

$$\int_{|x-x_\varepsilon^{(k)}|=\lambda_\varepsilon^{(k)}} |\nabla h_\varepsilon| \mathrm{d}s \leq (\pi\lambda_\varepsilon^{(k)})^{1/2} \left\{ \int_{|x-x_\varepsilon^{(k)}|=\lambda_\varepsilon^{(k)}} |\nabla h_\varepsilon|^2 \mathrm{d}s \right\}^{1/2} = o(1). \tag{6.16}$$

Now, integrate(2.4) over $Q^+ \setminus \cup_{k\in I_\varepsilon} B_{\lambda_\varepsilon^{(k)}}(x_\varepsilon^{(k)})$ to get, according to (6.16) and Lemma 12,

$$\int_{\Gamma_\varepsilon} \frac{\partial h_\varepsilon}{\partial \nu} \mathrm{d}s = 2\pi d + \sum_{k\in I_\varepsilon} \int_{|x-x_\varepsilon^{(k)}|=\lambda_\varepsilon^{(k)}} \frac{\partial h_\varepsilon}{\partial \nu} \frac{\mathrm{d}s}{\rho_\varepsilon^2} = 2\pi d - o(1) \text{ when } \varepsilon \to 0,$$

where I'_ε denotes the subset of indexes $k \in I_\varepsilon$ such that $B_{\lambda_\varepsilon^{(k)}}(x_\varepsilon^{(k)}) \cap Q^+ \neq \emptyset$, and $\Gamma_\varepsilon = \partial\Omega \setminus \cup_{k\in I'_\varepsilon} B_{\lambda_\varepsilon^{(k)}}(x_\varepsilon^{(k)})$. Therefore, there is $x_\varepsilon^* \in \Gamma_\varepsilon$ such that $\frac{\partial h_\varepsilon}{\partial \nu}(x_\varepsilon^*) > 0$. Similarly, we can show that on $\gamma_\varepsilon = \partial\omega \setminus \cup_{k\in I_\varepsilon \setminus I'_\varepsilon} B_{\lambda_\varepsilon^{(k)}}(x_\varepsilon^{(k)})$ there is x_ε^{**} such that $\frac{\partial h_\varepsilon}{\partial \nu}(x_\varepsilon^{**}) > 0$. $\square$

7 Inductive proof of Theorem 3

Fix $\Lambda > 0$ and an integer $d > 0$ such that $\Lambda > I_0(d, A)$, and let $æ_0$ be the greatest integer such that

$$I_0(d, A) + \pi æ_0 < \Lambda.$$

Clearly, $æ_0 \geq 0$. In this Section we show that for small ε the infimum in problem (1.7) is always attained, provided integers p, q satisfy

$$æ(p, q) \leq æ_0 \quad \text{and} \quad p \leq d,\ q \leq d, \tag{7.1}$$

where $æ(p, q) = |d - q| + |d - p|$.

Proposition 24. *Given an integer $K \leq æ_0$. Let $p \leq q$, $q \leq d$ be integers such that $æ(p, q) \leq K$. Then, for sufficiently small ε the infimum in problem (1.7) is always attained and*

$$m_\varepsilon(p, q, d) \leq m_\varepsilon(l, m, d) + \pi((l-p) + (m-q)) \quad \textit{when} \quad p \leq l \leq d,\ q \leq m \leq d. \tag{7.2}$$

Moreover, the inequality in (7.2) is strict unless $l = p$ and $m = q$.

Proof. The proof is by induction on K. The basis of induction (K=0) is established in Section 4 (cf. Lemma 17). The demonstration of the induction step relies on the following Lemma, whose proof is in the end of this section.

Lemma 25. *Assume integers p and q satisfy (7.1), and for $\varepsilon < \varepsilon_5$, $\varepsilon_5 > 0$, there exists a minimizer u_ε of problem (1.7) whose GL energy $E_\varepsilon(u_\varepsilon)$ satisfies the bound (6.1). Then for any $\varepsilon < \min\{\varepsilon_4, \varepsilon_5\}$ (where ε_4 is as in Lemma 23) there exists $v_\varepsilon \in \mathcal{J}_{p(q-1)}$ such that* $\operatorname{abdeg}(v_\varepsilon) \in (d - 1/2, d + 1/2)$ *and*

$$E_\varepsilon(v_\varepsilon) < m_\varepsilon(p, q, d) + \pi. \tag{7.3}$$

Similarly, in $\mathcal{J}_{(p-1)q}$ there exists a testing map (still denoted v_ε) satisfying (7.3), and such that $\operatorname{abdeg}(v_\varepsilon) \in (d - 1/2, d + 1/2)$.

In view of Lemma 25, to prove the claim of Proposition 24 for $K + 1$ in place of K it suffices to show that $m_\varepsilon(p, q - 1, d)$ is always attained for sufficiently small ε whenever $p \leq d$, $q \leq d$ and $æ(p, q) = K$ (the attainability of $m_\varepsilon(p-1, q, d)$ is proved similarly). Let u be a weak H^1-limit of a minimizing sequence $(u^{(k)})$. According to Lemma 25, such a minimizing sequence exists, moreover, this lemma, with the induction hypothesis, imply

$$\limsup_{k\to\infty} E_\varepsilon(u^{(k)}) < m_\varepsilon(l', m', d) + \pi(l' - p) + \pi(m' - (q - 1)), \tag{7.4}$$

where $p \leq l' \leq d$, $q - 1 \leq m' \leq d$ and $æ(l', m') \leq K$. We know that $\operatorname{abdeg}(u) = \lim_{k\to\infty} \operatorname{abdeg}(u^{(k)}) \in (d - 1/2, d + 1/2)$ when $\varepsilon < \varepsilon_0$ (where $\varepsilon_0 = \varepsilon_0(\Lambda)$ is as in Proposition 2), hence u is a solution of the GL equation (1.1) (see arguments in Lemma 17). Therefore, if we write $\liminf_{k\to\infty} E_\varepsilon(u^{(k)}) \leq \limsup_{k\to\infty} E_\varepsilon(u^{(k)})$ and apply successively Lemma 9 and Lemma 16 to the left hand side, we get by using (7.4) with $l' = m' = d$ that for ε sufficiently small

$$I_0(d, A) - \frac{\pi}{2} + \pi æ(l, m) + \pi(|l - p| + |m - (q - 1)|) \leq m_\varepsilon(d, d, d) + \pi æ(p, q - 1),$$

where $l = \deg(u, \partial\omega)$, $m = \deg(u, \partial\Omega)$. Thanks to Lemma 14 $m_\varepsilon(d, d, d) \leq I_0(d, A)$, thus

$$|l - d| + |l - p| \leq |p - d| + 1/2 \text{ and } |m - d| + |m - (q - 1)| \leq |(q - 1) - d| + 1/2.$$

Since l and m are integers, it follows that $p \leq l \leq d$, $q - 1 \leq m \leq d$. Now, assuming $l \neq p$ or $m \neq q - 1$, we use Lemma 9 and (7.4) with $l' = l$, $m' = m$ to obtain the following

$$E_\varepsilon(u) + \pi(l - p) \leq \liminf_{k\to\infty} E_\varepsilon(u^{(k)}) < m_\varepsilon(l, m, d) + \pi(l - p) + \pi(m - (q - 1)).$$

On the other hand, $u \in \mathcal{J}_{lm}$ and $\mathrm{abdeg}(u) \in [d-1/2, d+1/2]$. Hence $E_\varepsilon(u) \geq m_\varepsilon(l,m,d)$, which is a contradiction. Therefore $l = p$ and $m = q-1$, i.e. u is an admissible testing map in problem (1.7) and thus the infimum $m_\varepsilon(p, q-1, d)$ is always attained. □

Proof of Lemma 25. For simplicity we drop subscript ε. The underlying idea is to modify the minimizer u of (1.7) in a neighborhood of $\partial\Omega$ as in Proposition 18 (see Section 5). In general u is with zeros now, thus the arguments need to be more sophisticated. Loosely speaking, we construct a testing map v with an additional "vortex" located "near" x^*, where x^* is a point on $\partial\Omega$ such that $\frac{\partial h}{\partial \nu}(x^*) > 0$ (cf. Lemma 23).

Step 1: Domain decomposition. Let $1-\delta$, where $\delta > 0$, be a regular value of h (thanks to Stard's lemma this holds for almost all δ). Consider the subdomain of A where $h > 1-\delta$. There is a (unique) connected component D_δ of this subdomain, such that $\partial D_\delta \supset \partial\Omega$. Since $h(\partial\Omega) = 1 > h(\partial\omega)$, when δ is sufficiently small the boundary of D_δ contains a connected component $\Gamma_\delta \neq \partial\Omega$ enclosing ω. According to Lemma 23, we can choose δ small enough $\delta < \delta_0$ ($\delta_0 > 0$) so that the domain enclosed by Γ_δ and $\partial\Omega$ lies away from the contour $\mathcal{L}$, i.e. Γ_δ also encloses $\mathcal{L}$, moreover if $\rho(x)(= |u(x)|)$ vanishes at a point x of this domain, then $h(x) > 1$. Therefore, the minimum of h over the closure of the forementioned domain cannot be attained at any interior point, otherwise h satisfies $\mathrm{div}(\frac{1}{\rho^2} h) = 0$ in a neighborhood of this point, which is impossible. In other words, $h > 1-\delta$ in the domain enclosed by Γ_δ and $\partial\Omega$, i.e. this domain coincides with D_δ. Thus, the boundary of D_δ consists of exactly two connected components $\partial\Omega$ and Γ_δ. Also, possibly choosing smaller δ_0, we have that the set $P = \{x \in D_\delta;\, h(x) \geq 1\}$ is independent of δ (recall that $\delta < \delta_0$ and $1-\delta$ is a regular value of h). Indeed, consider the set $S_\delta = \{x \in A; h(x) > \alpha\} \cap D_\delta$ ($0 < \delta < \delta_0$), where α is a regular value of h and $1 < \alpha < 9/8$. S_δ consists of a finite number $n(\delta)$ of connected components. Since $D_\delta \supset D_{\delta'}$ if $\delta > \delta'$, the function $n(\delta)$ is nondecreasing, hence $n(\delta) = \lim_{\delta' \to 0} n(\delta')$ when $0 < \delta < \delta_0$ (for some $\delta_0 > 0$) and $S_\delta = S_{\delta'}$ if $\delta, \delta' \in (0, \delta_0)$. It follows that $1-\delta < h < \alpha$ in $D_\delta \setminus D_{\delta'}$ when $0 < \delta' < \delta < \delta_0$. For such δ and δ' the function h satisfies $\mathrm{div}(\frac{1}{\rho^2} h) = 0$ in $D_\delta \setminus D_{\delta'}$ (by Lemma 23) while $h < 1$ on the boundary of $D_\delta \setminus D_{\delta'}$, hence $h < 1$ in $D_\delta \setminus D_{\delta'}$. We see now that $\{x \in D_\delta; h(x) \geq 1\} = P := \cap_{\delta' < \delta_0} D_{\delta'}$, when $0 < \delta < \delta_0$, as required.

Thus we have, for $\delta < \delta_0$

$$d' := \frac{1}{2\pi}\int_{\Gamma_\delta} \frac{\partial h}{\partial \nu}\frac{\mathrm{d}s}{\rho^2} = \frac{1}{2\pi}\int_{\Gamma_\delta} \frac{u}{|u|} \times \frac{\partial}{\partial \tau}\frac{u}{|u|}\mathrm{d}s = \deg(u, \Gamma_\delta) > 0$$

($1-\delta$ is a regular value of h and $h > 1-\delta$ in D_δ) and the integer d' is independent of δ. Therefore u admits the representation $u = \rho e^{id'\theta}$ in $G_\delta = D_\delta \setminus P$, where $\theta : G_\delta \to \mathbb{R} \setminus 2\pi\mathbb{Z}$ is a smooth function and $\rho > 0$ in $\overline{G}_\delta$.

Without loss of generality we can assume that $u(x^*) = 1$. Since $\nabla h = -d'\rho^2\nabla^\perp\theta$ in G_δ and $\frac{\partial h}{\partial \nu}(x^*) > 0$, the map $x \to (h, \theta)$ from a neighborhood G'_δ of x^* onto its image $h(G'_\delta)$ is a C^1-diffemorphism. Choosing δ small enough, we can assume that G'_δ is defined by

$$x \in G'_\delta \iff x \in G_\delta,\ 1-\delta < h(x) < 1,\ \theta(x) \in (-\delta, \delta)(\operatorname{mod} 2\pi\mathbb{Z}).$$

Now we have $A = G'_\delta \cup G''_\delta \cup (A \setminus G_\delta)$ (see Fig. 1), where $G''_\delta = G_\delta \setminus G'_\delta$.

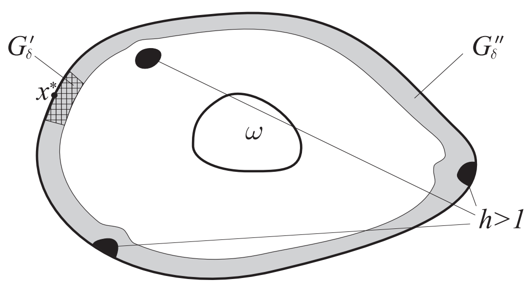

Figure 1: Domain decomposition.

Step 2: Construction of the testing map. We seek the tasting map v in the form

$$v = \begin{cases} u & \text{in } A \setminus G_\delta, \\ \rho w_t & \text{in } G_\delta, \end{cases} \tag{7.5}$$

with (unknown for the moment) $w_t = w_t(h(x), \theta(x))$. Impose the following boundary conditions on ∂G_δ,

$$w_t = \mathrm{e}^{id'\theta}\frac{\mathrm{e}^{-i\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-i\theta}(1 - t\varphi(\theta)) - 1} \quad \text{on} \quad \partial G_\delta \setminus \Gamma_\delta \tag{7.6}$$

$$w_t = \mathrm{e}^{id'\theta} \quad \text{on} \quad \Gamma_\delta, \tag{7.7}$$

where $0 \le \varphi \le 1$ is a smooth 2π-periodic cut-off function such that $\varphi(\theta) = 1$, when $\theta \in (-\delta/2, \delta/2) \pmod{2\pi\mathbb{Z}}$ and $\varphi(\theta) = 0$ if $\theta \notin (-\delta, \delta) + 2\pi\mathbb{Z}$. It is easy to see that if w_t (considered as a function of h, θ) satisfies (7.6) and (7.7), when $h = 1$ and $h = 1 - \delta$, respectively, and is a smooth $2\pi-$periodic in θ map defined in the strip $1-\delta \le h \le 1$ then (7.5) defines for $0 < t < 1$ a map $v \in H^1(A; \mathbb{R}^2)$ such that $|v| = 1$ on ∂A and

$$\deg(v, \partial\Omega) = q - 1, \quad \deg(v, \partial\omega) = p. \tag{7.8}$$

Expand the right hand side of (7.6) into the series

$$\mathrm{e}^{id'\theta} \frac{\mathrm{e}^{-i\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-i\theta}(1 - t\varphi(\theta)) - 1} = (1 - tb_{-1}(t))\mathrm{e}^{id'\theta} + t \sum_{k \ne -1} b_k(t) \mathrm{e}^{-i(k-d'+1)\theta}, \tag{7.9}$$

and set

$$w_t(h, \theta) = (1 - tb_{-1}(t) f_{-1}(h))\mathrm{e}^{id'\theta} + t \sum_{k \ne -1} b_k(t) f_k(h) \mathrm{e}^{-i(k-d'+1)\theta}, \tag{7.10}$$

where functions f_k are defined by (5.19) with $k_\pm = \pm\dfrac{1}{d'}\sqrt{(k - d' + 1)^2 + \lambda - d'^2}$. The positive parameters $t < 1$ and $\lambda \ge 2d'^2$ are to be specified later on. In what concerns the coefficients $b_k(t)$, we have

$$|b_k(t) - c_k(t)| \le C(1 + |k|)^{-n}, \ \forall n > 0, \tag{7.11}$$

where $C = C(n)$ is independent of t, and $c_k = (t - 2)(1 - t)^k$ for $k \ge 0$, $c_{-1} = 1$, $c_k = 0$ for $k < -1$. The estimate (7.11) is obtained in a standard way, by comparing the Fourier coefficients in (7.9) with those of $\mathrm{e}^{id'\theta}(\mathrm{e}^{-i\theta} - (1 - t))/(\mathrm{e}^{-i\theta}(1 - t) - 1)$.

Step 3: Verification of (7.3). Thanks to Lemma 19 we have (since $|w_t| = 1$ on ∂G_δ due to (7.6, 7.7))

$$E_\varepsilon(v) = E_\varepsilon(u) + L_\varepsilon^{(d')}(w_t, G_\delta)$$

where the functional $L_\varepsilon^{(d')}(w)$ is defined as in (5.8). Let us show, that for sufficiently small t

$$L_\varepsilon^{(d')}(w_t, G_\delta) < \pi. \tag{7.12}$$

To this end, like in the proof of Proposition 18, consider the quadratic functional

$$M'_\lambda(w_t) = \frac{1}{2} \int_{G_\delta} (d'^2 |\partial_r w_t|^2 + |\partial_\theta w_t|^2 + \lambda |w_t - \mathrm{e}^{i\theta}|^2 - d'^2 |w_t|^2) \rho^2 |\nabla\theta|^2 \mathrm{d}x. \tag{7.13}$$

(One can actually show that w_t minimizes functional (7.13) under the boundary conditions (7.6), (7.7).) We have, since $\nabla h = -d'\rho^2\nabla^\perp\theta$ in G_δ and $\rho \le 1$ in A,

$$\int_{G_\delta} \rho^2 |\nabla w_t|^2 \mathrm{d}x \le \int_{G_\delta} (d'^2 |\partial_r w_t|^2 + |\partial_\theta w_t|^2)\rho^2 |\nabla\theta|^2 \mathrm{d}x. \tag{7.14}$$

Moreover, if we put

$$\lambda = \max\left\{ \frac{9}{2\varepsilon^2 \min_{\overline{G'_\delta}} |\nabla\theta|^2}, 2d'^2 \right\},$$

under the additional assumption that $|w_t| \le 2$ in G'_δ, the following pointwise inequality $2\varepsilon^2\lambda|w_t - \mathrm{e}^{i\theta}|^2|\nabla\theta|^2 \ge \rho^2(|w_t|^2-1)^2$ holds in G'_δ (see the proof of Proposition 18). Thus,

$$L_\varepsilon^{(d')}(w_t, G_\delta) \le M'_\lambda(w_t) + \frac{1}{4\varepsilon^2}\int_{G''_\delta} \rho^4(|w_t|^2-1)^2 \mathrm{d}x. \tag{7.15}$$

To demonstrate (7.12) we first note that

$$M'_\lambda(w_t) = \frac{t^2\pi}{d'} \sum_{k=-\infty}^{\infty} |b_k(t)|^2 \Phi'_k(f_k), \tag{7.16}$$

where functionals Φ'_k are defined as functionals Φ_k in (5.18) with d' in place of d. The representation (7.16) is analogous to that in (5.17) while its justification differs because of the fact that ∇h vanishes at least at some points of the boundary of G_δ (and possibly somewhere in G''_δ). We rely on the following Lemma, which implies directly (7.16).

Lemma 26. *Let $f, g \in C^1([1-\delta, 1]; \mathbb{C})$ then for all integers n, m*

$$\int_{G_\delta} f(h)\mathrm{e}^{in\theta}\overline{g(h)\mathrm{e}^{im\theta}}\,\rho^2|\nabla\theta|^2 \mathrm{d}x = \begin{cases} 0, & \text{if} \quad n \ne m \\ \dfrac{2\pi}{d'}\displaystyle\int_{1-\delta}^1 f(s)\bar{g}(s)\mathrm{d}s, & \text{if} \quad n = m. \end{cases}$$

Proof. By virtue of the pointwise equalities $\nabla h \cdot \nabla\theta = 0$ and $\mathrm{div}(\rho^2\theta) = 0$ in G_δ, for any regular values α, β of h such that $1-\delta \le \alpha < \beta \le 1$ and any integer $n \ne m$,

$$\begin{aligned} \int_{\alpha<h<\beta} f\mathrm{e}^{in\theta}\overline{g\mathrm{e}^{im\theta}}\,\rho^2|\nabla\theta|^2\mathrm{d}x &= \frac{-i}{n-m}\int_{\alpha<h<\beta} \nabla\theta\cdot\nabla\mathrm{e}^{i(n-m)\theta} f\bar{g}\rho^2 \mathrm{d}x \\ &= \frac{i}{n-m}\int_{\alpha<h<\beta} \mathrm{div}(\rho^2\nabla\theta) f\bar{g}\mathrm{e}^{i(n-m)\theta}\mathrm{d}x \\ &\quad + \frac{i}{n-m}\int_{\alpha<h<\beta} \nabla\theta\cdot\nabla h(f'\bar{g} + f\bar{g}')\mathrm{e}^{i(n-m)\theta}\rho^2\mathrm{d}x = 0, \end{aligned} \tag{7.17}$$

where all integrals are understood over subsets of G_δ. If $n = m$ we set $F(h) = \int_\alpha^r f(s)\bar{g}(s)ds$ then, since $|\nabla h| = d'\rho^2|\nabla\theta|$

$$\int_{\alpha<h<\beta} f(h)\bar{g}(h)\rho^2|\nabla\theta|^2\mathrm{d}x = \frac{1}{d'^2}\int_{\alpha<h<\beta}\nabla(F(h))\cdot\nabla h\frac{\mathrm{d}x}{\rho^2} = \frac{1}{d'^2}F(\beta)\int_{h=\beta}\frac{\partial h}{\partial\nu}\frac{\mathrm{d}s}{\rho^2}$$
$$-\frac{1}{d'^2}\int_{\alpha<h<\beta}\mathrm{div}(\frac{1}{\rho^2}\nabla h)F(h)\mathrm{d}x = \frac{2\pi}{d'}\int_\alpha^\beta f(s)\bar{g}(s)\mathrm{d}s, \quad (7.18)$$

here we have also used the fact that $\mathrm{div}(\frac{1}{\rho^2}\nabla h) = 0$ in G_δ. The statement of the Lemma is then obtained by passing to the limits $\alpha \to 1-\delta$ and $\beta \to 1$ in (7.17, 7.18). □

By using (7.11) in (7.16) we compute

$$M'_\lambda(w_t) = \frac{t^2\pi}{d'}\sum_{k=-\infty}^{\infty}|c_k(t)|^2\Phi'_k(f_k) + O(t^2) = \pi((1-t)^2-1)^2\sum_{k=0}^{\infty}k(1-t)^{2k}$$
$$+ 2\pi t^2(\lambda - d'^2)\sum_{k=1}^{\infty}\frac{(1-t)^{2k}}{k} + O(t^2) = \pi - 2\pi t + o(t). \quad (7.19)$$

Explicit, but tedious computations (left to the reader), show also that

$$|w_t - \mathrm{e}^{id'\theta}| \le Ct \text{ when } \theta \notin (-\alpha,\alpha) + 2\pi\mathbb{Z}, \quad (7.20)$$

for any $0 < \alpha < 2\pi$, where C is independent of t. From (7.20), we see the second term in (7.15) is of order $O(t^2)$ as $t \to 0$. Combined with (7.19) this proves (7.12).

Final step. The bound (7.15) is shown assuming that $|w_t| < 2$ in G''_δ. Note, that this can always be achieved by replacing w_t by $\tilde{w}_t := w_t\min\{1, 2/|w_t|\}$, and this change increases neither the first term in (7.15) nor the second one. Thus in order to complete the proof of the lemma, we need to show only that the map v defined by (7.5) satisfies $d - 1/2 \le \mathrm{abdeg}(v) \le d + 1/2$ when t is chosen sufficiently small. Indeed, due to (7.20) w_t weakly H^1-converges to $\mathrm{e}^{id'\theta}$. Therefore the norm $\|u - v\|_{L^2(A)}$ tends to 0 when $t \to 0$. Then, according to Lemma 10, for small t $\mathrm{abdeg}(v)$ is close to $\mathrm{abdeg}(u)$, while $d - 1/2 < \mathrm{abdeg}(u) < d + 1/2$ and we are done. □

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Appendix A

Here is a simple example of Möbius (Blashke) test map that illustrates an important property of nearboundary vortices. Namely, each such vortex appears in pair with a "ghost" antivortex (vortex with opposite sign) coming from outside the domain. For simplicity consider the minimization problem of (1.2) with $A = B_1$, boundary condition $|u| = 1$ on ∂B_1 and imposed degree one on the boundary ∂B_1. It is shown

in [11] that the infimum in this problem is not attained and the behavior of the minimizing sequence is described by

$$v_\varepsilon(z) = \frac{\bar{\zeta}_\varepsilon}{|\zeta_\varepsilon|} \frac{z-\zeta_\varepsilon}{\bar{\zeta}_\varepsilon z - 1} = \frac{1}{|\zeta_\varepsilon|} \frac{z-\zeta_\varepsilon}{z-(\zeta_\varepsilon + o(1))},$$

where $\zeta_\varepsilon \in B_1$, $|\zeta_\varepsilon| \to 1$ as $\varepsilon \to 0$. We assume that $\mathrm{dist}(\zeta_\varepsilon, \partial B_1) = o(\varepsilon)$. Clearly this map has single zero of degree one at $z = \zeta_\varepsilon$. Introduce $\zeta_\varepsilon^* = 1/\bar{\zeta}_\varepsilon$, and write this map as

$$v_\varepsilon(z) = \frac{1}{|\zeta_\varepsilon|} \frac{|z-\zeta_\varepsilon|}{|z-\zeta_\varepsilon^*|} \frac{z-\zeta_\varepsilon}{|z-\zeta_\varepsilon|} \left(\frac{z-\zeta_\varepsilon^*}{|z-\zeta_\varepsilon^*|} \right)^{-1} = \omega_\varepsilon(z) \frac{z-\zeta_\varepsilon}{|z-\zeta_\varepsilon|} \frac{\overline{z-\zeta_\varepsilon^*}}{|z-\zeta_\varepsilon^*|}, \tag{A.1}$$

where $\omega_\varepsilon(z)$ is a real-valued function. Then, the last factor in (A.1) corresponds to a "ghost" vortex with the center at $\zeta_\varepsilon^* \notin B_1$. The complex conjugation means this vortex has degree -1 which is why it is called antivortex. Finally, $\omega_\varepsilon(z) = 1 + o(1)$ outside the disk of radius ε with center $z = \zeta_\varepsilon$. These observations show superposition of vortex and antivortex has almost no energy away from the core, whereas, the contribution in the energy from outside the core of an inner vortex is known to grow as $\log(1/\varepsilon)$.

Appendix B

Let us show that, for any integers p, q, d, the set of admissible testing maps in problem (1.7) is not empty and

$$m_\varepsilon(p,q,d) \le I_0(d,A) + \pi|q-d| + \pi|p-d|. \tag{B.1}$$

To this end chose two sequences $(x_1^{(k)}), (x_2^{(k)}) \subset A$ such that $x_1^{(k)} \to \partial\Omega$, $x_2^{(k)} \to \partial\omega$. Consider solutions $\rho_1^{(k)}$, $\rho_2^{(k)}$ of the boundary value problems

$$\begin{cases} \Delta\rho_1^{(k)} = 2\pi|d-q|\delta(x-x_1^{(k)}) \text{ in } A \\ \rho_1^{(k)} = 0 \quad \text{on } \partial\Omega, \\ \rho_1^{(k)} = \text{Const} \quad \text{on } \partial\omega \\ \displaystyle\int_{\partial\Omega} \frac{\partial\rho_1^{(k)}}{\partial\nu} d\sigma = 2\pi|d-q|, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\rho_2^{(k)} = 2\pi|d-p|\delta(x-x_2^{(k)}) \text{ in } A \\ \rho_2^{(k)} = 0 \quad \text{on } \partial\omega, \\ \rho_2^{(k)} = \text{Const} \quad \text{on } \partial\Omega \\ \displaystyle\int_{\partial\omega} \frac{\partial\rho_2^{(k)}}{\partial\nu} d\sigma = 2\pi|d-p|. \end{cases}$$

One can show that, for any neighborhoods U and V of $\partial\Omega$ and $\partial\omega$ in A,

$$\rho_1^{(k)} \to 0 \text{ in } C^1(A \setminus U) \quad \text{and} \quad \rho_2^{(k)} \to 0 \text{ in } C^1(A \setminus V). \tag{B.2}$$

There exist harmonic conjugates

$$\begin{aligned}\psi_1^{(k)} &: A \setminus \{x_1^{(k)}\} \to \mathbb{R} \setminus 2\pi|d-q|\mathbb{Z},\\ \psi_2^{(k)} &: A \setminus \{x_2^{(k)}\} \to \mathbb{R} \setminus 2\pi|d-p|\mathbb{Z},\end{aligned}$$

(i.e. $\nabla\psi_j^{(k)} = \nabla^\perp\rho_j^{(k)}$, $j=1,2$) so that $e^{\rho_j^{(k)}+i\psi_j^{(k)}}$ are harmonic functions in A. By using the pointwise equality $\frac{1}{2}|\nabla u|^2 = \partial_{x_1}u \times \partial_{x_2}u + \frac{1}{4}|\partial_{\bar z}u|^2$ and integrating by parts we get

$$\frac{1}{2}\int_A e^{2\rho_1^{(k)}}(|\nabla\rho_1^{(k)}|^2 + |\nabla\psi_1^{(k)}|^2)\,\mathrm{d}x = \frac{1}{2}\int_A |\nabla e^{\rho_1^{(k)}+i\psi_1^{(k)}}|^2\,\mathrm{d}x = \pi|d-q|. \tag{B.3}$$

Similarly we have,

$$\frac{1}{2}\int_A e^{2\rho_2^{(k)}}(|\nabla\rho_2^{(k)}|^2 + |\nabla\psi_2^{(k)}|^2)\,\mathrm{d}x = \pi|d-p|. \tag{B.4}$$

Then we introduce

$$u^{(k)}(x) = u(x)e^{\rho_1^{(k)}(x)+i\,\mathrm{sgn}(q-d)\psi_1^{(k)}(x)}e^{\rho_2^{(k)}(x)+i\,\mathrm{sgn}(p-d)\psi_2^{(k)}(x)}e^{-\rho_1^{(k)}(\partial\omega)+V(x)(\rho_1^{(k)}(\partial\omega)-\rho_2^{(k)}(\partial\Omega))},$$

where u is a minimizer of problem (1.14) and V is the solution of (1.9). From (B.2-B.4) we derive that, for k sufficiently large, $u^{(k)}$ is an admissible testing map for problem (1.7), and that $E_\varepsilon(u^{(k)}) \to I_0(d,A) + \pi|q-d| + \pi|p-d|$ when $k \to \infty$.